Introduction to Topological Groups

An Introductory Course
from the Fourth Semester up
Qualification Module
Wahlpflichtbereich und Haupstudium

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Chapter 1
Topological groups

Topological groups have the algebraic structure of a group and the topological structure of a topological space and they are linked by the requirement that multiplication and inversion are continuous functions. Most infinite groups we encounter in any areas of mathematics are topological groups such as the group of $n \times n$ invertible matrices, the additive and multiplicative groups of the fields $\mathbb{R}$ and $\mathbb{C}$ and their subgroups such as for instance the multipicative group $\mathbb{S}^1$ of complex numbers of absolute value 1.

In this course we shall introduce the relevant concepts in order to be able to discuss topological groups and we shall develop their basic theory.

The prerequisites for the course are Linear Algebra I and II, Introduction to Algebra, Analysis I and II; it would help to have had Introduction to Complex Variables but that is not absolutly necessary, and it would be indeed helpful to have had Introduction to Topology.

Homogeneous topological spaces

We shall shortly repeat in a formal fashion the definition of a topological group given in the introductory comments. One might ask the following elementary question:

Given a topologial space X, can we find a group multiplication and inversion on X such that we obtain a topological group?

We know the answer is "yes" for the open half-line $]0, \infty[$ in the euclidean space $\mathbb{R}$, because ordinary multiplication of real numbers makes this space into a topological group. The answer is "no" for the closed half-line $[0, \infty[$. But why? The unit ball $B^n \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, $n > 0$, in euclidean n space cannot be a topological group as we shall see presently. The surface $\mathbb{S}^n \stackrel{\text{def}}{=} \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ of the $n+1$-ball B^{n+1}, $n \geq 0$ is called the n-sphere. Can it be made into a topological group? We know that the answer is "yes" for $n = 1$ because the multiplication of complex numbers of absolute value 1 makes $\mathbb{S}^1$ into a topological groups. What is the situation for $n > 1$? For instance, can $\mathbb{S}^2$ be given the structure of a topological group? [This is not an easy question. See, for instance K.H.Hofmann and S.A.Morris, The Structure of Compact Groups, Berlin, 1998, Corollary 9.59(iv), p.486.]

We cannot hope to find necessary and sufficient conditions in general. But even necessary conditions would be a welcome help towards answering the question. One such condition, as we shall see as first order of business is the property of homogeneity.

Definition 1.1. A topological space X is called *homogeneous* if for $(x, y) \in X \times X$ there is a homeomorphism $f: X \to X$ such that $f(x) = y$.

Recall that a group G is said to *act* on a set X if there is a function $(g, x) \mapsto g{\cdot}x: G \times X$ such that $1{\cdot}x = x$ for all x and $g{\cdot}(h{\cdot}x) = (gh){\cdot}x$ for all $g, h \in G$ and $x \in X$. A group always acts upon itself by each of the following operations:

(i) $g{\cdot}x = gx$ (left multiplication),

(ii) $g{\cdot}x = xg^{-1}$, (right multiplication),

(iii) $g{\cdot}x = gxg^{-1}$, (conjugation).

We say that G acts transitively, if the action has only one *orbit*, i.e. $X = G{\cdot}x$ for some (and then any) $x \in X$.[1] The action of a group on itself by multiplication is transitive.

Now we can say that X is homogeneous if the group of all homeomorphisms of X operates transitively on X.

Proposition 1.2. *Let G be a group acting on a topological space X such that the function $x \mapsto g{\cdot}x: X \to X$ is continuous for each $g \in G$. If G acts transitively, then X is homogeneous.*

The full homeomorphism group of a space X acts on the space by evaluation $(f, x) \mapsto f(x)$ such that $x \mapsto f(x)$ is continuous.

Lemma 1.3. *Assume that every point x of a space X has a neighborhood U such that for each $u \in U$ there is a homeomorphism f of X such that $f(x) = u$.*

Then each orbit of the homeomorphism group G of X is open. In particular, if X is connected, then X is homogeneous.

Proof. Assume $x \in G{\cdot}y$. Let U be as in the statement of the Lemma, and consider $u \in U$. Then there is an $f \in G$ such that $x = f(y) = f{\cdot}y$ and a $g \in G$ such that $u = g{\cdot}x$. Then $u = (g \circ f){\cdot}y \in G{\cdot}y$ and thus $G{\cdot}y$ is open.

The orbits thus form a partition of X into open equivalence classes. In such a case each equivalence class, being the complement of the union of all other equivalence classes is also closed. If X is connected X is the only orbit of the action. □

For the followiong Lemma recall that the vector space $\mathbb{R}^n$ with the euclidean norm given by $\|x\|^2 = \sum_{m=1}^{n} x_m^2$ for $x \in \mathbb{R}^n$ is a normed vector space.

Lemma 1.4. (i) *In a normed vector space E, the closed unit ball $B = \{x \in E : \|x\| \leq 1\}$ with center 0 and boundary $D = \{x \in E : \|x\| = 1\}$ has the property that for each u in the interior of B there is a homeomorphism f_u of B such that $f_u(0) = u$ and f_u leaves every point $d \in D$ fixed.*

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[1] German readers: This a good example of the semantic difference of “some” and “any” for which there is, technically, only one German word. Im gegenwärtigen Fall ist „. . . für ein (und daher jedes) $x \in X$“ eine brauchbare Übersetzung.

(ii) Let X be a topological space and B a closed subspace homeomorphic to a closed unit ball of a finite dimensional normed vector space such that the interior U of B is mapped onto the interior of the unit ball. Then for $(x, y) \in U \times U$ there is a homeomorphism f of X such that $f(x) = y$.

Proof . (i) For $0 \leq s$ and $0 \leq t < 1$, set $\sigma(s) = (1+s^2)^{-1/2}$ and $\tau(t) = (1-t^2)^{-1/2}$ and define

$$\varphi: B \setminus D \to E, \ \varphi(x) = \tau(\|x\|)\cdot x. \tag{1}$$

Then $y \in E$ implies $\varphi^{-1}(y) = \sigma(\|y\|)(y)$ and $\|x\| = \sigma(\|\varphi(x)\|)$. Now let $u \in B \setminus D$. Define $T_u: E \to E$ by $T_u(x) = x + \varphi(u)$. Finally, define $f_u: B \to B$ by

$$f_u(x) = \begin{cases} \varphi^{-1}T_u\varphi(x) & \text{if } \|x\| < 1, \\ x & \text{if } \|x\| = 1. \end{cases}$$

Then

$$f_u^{-1}(x) = \begin{cases} \varphi^{-1}T_u^{-1}\varphi(x) & \text{if } \|x\| < 1, \\ x & \text{if } \|x\| = 1. \end{cases}$$

So f_u and $f_u^{-1} = f_{-u}$ are inverses of each other, are continuous on $B \setminus D$, and fix the boundary D of B pointwise; moreover $f_u(0) = u$. It remains to show that f_u and f_u^{-1} are continuous in each point d of D; obviously it suffices to prove this for f_u. Since $f_u(x) = x$ for $x \in D$, we must show that $\|d - x\| \to 0$, $\|x\| < 1$ implies $\|d - f_u(x)\| \to 0$. We now assume $\|d-x\| \to 0$. Then $\|x\| \to 1$ and $\|d - \|x\|^{-1}\cdot x\| \leq \|d-x\| + (1-\|x\|^{-1})\|x\| \to 0$. Since $\|d - f_u(x)\| \leq \|d - \|x\|^{-1}\cdot x\| + \|\|x\|^{-1}\cdot x - f_u(x)\|$ it suffices to verify $\|\|x\|^{-1}\cdot x - f_u(x)\| \to 0$. Now $T_u\varphi(x) = \varphi(x) + \varphi(u)$, and so

$$f_u(x) = \varphi^{-1}\big(\varphi(x) + \varphi(u)\big) = \alpha(x)\cdot\big(\varphi(x) - \varphi(u)\big)$$

where $\alpha(x) = (1 + \|\varphi(x) + \varphi(u)\|^2)^{-1/2} = \|\varphi(x)\|^{-1}\cdot\beta(x)$ with

$$\beta(x) \to 1 \text{ for } \varphi(x) \to \infty. \tag{2}$$

This gives us

$$f_u(x) = \beta(x)\|\varphi(x)\|^{-1}\varphi(x) - \beta(x)\varphi(x)^{-1}\cdot\varphi(u). \tag{3}$$

We note

$$\nu(x) \stackrel{\text{def}}{=} \beta(x)\varphi(x)^{-1}\cdot\varphi(u) \to 0 \text{ for } \|x\| \to 1. \tag{4}$$

From (1) we get $\|\varphi(x)\|^{-1}\cdot\varphi(x) = (\tau(\|x\|)\cdot\|x\|)^{-1}\tau(\|x\|)\cdot x = \|x\|^{-1}\cdot x$. Thus

$$\|\|x\|^{-1}\cdot x - f_u(x)\| = \big(1 - \beta(x)\big)\|x\|^{-1}\cdot x + \nu(x) \to 0$$

for $x \to d$ by (2), (3) and (4). This is what we had to show. We notice that if x and y are two points in the open unit ball then $f \stackrel{\text{def}}{=} f_y \circ f_x^{-1}$ is a homeomorphism of B fixing the boundary pointwise and satisfying $f(x) = y$.

(ii) By (i) above, given $(x, y) \in U \times U$ there is a bijection of X which agrees with the identity map on the closed subset $X \setminus U$ and is a homeomorphism on

U such that $x = f(y)$ It is now an exercise to verify that that f and f^{-1} are continuous functions $X \to X$. □

Exercise E1.1. Check and verify the details of the preceding proof. □

Definition 1.5. A *topological manifold* is a topological space each point of which has an open neighborhood which is homeomorphic to $\mathbb{R}^n$ for some n. □

An open ball of a finite dimensional normed real vector space is homeomorphic to $\mathbb{R}^n$.

Exercise E1.2. Show that

(i) *every open subset of* $\mathbb{R}^n$ *is a topological manifold.*

(ii) *Every sphere* $\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$ *is a compact regular topological manifold.*

(iii) *Every finite product of regular topological manifolds is a regular topological manifold.* □

In particular, a torus $\mathbb{S}^1 \times \mathbb{S}^1$ is a topological manifold. Note that every discrete space is a regular topological manifold.

Proposition 1.6. *A connected Hausdorff topological manifold is homogeneous.*

Proof . Let $f\colon \mathbb{R}^n \to X$ be a homeomorphism onto an open neighborhood of $x \stackrel{\text{def}}{=} f(0)$ in the manifold X. Equip $\mathbb{R}^n$ with the euclidean norm and set $U = \{v \in \mathbb{R}^n : \|v\| < 1\}$ and $B = \{v \in \mathbb{R}^n : \|v\| \le 1\}$. Since f is a homeomorphism, $f(U)$ is open in $f(\mathbb{R}^n)$ and thus is a neighborhood of x. Hence $f(B)$ is a compact neighborhood of x that is homeomorphic to a closed euclidean unit ball. Since X is Hausdorff, $f(B)$ is closed in X.

Now we apply Lemma 1.4 and Lemma 1.3 and conclude that the connectedness of X implies that X is homogeneous. □

Attention: There are some delicate points that illustrate the necessity of restricting our attention to Hausdorff manifolds. Consider the following

Example. Let $X \subseteq \mathbb{R}^2$ be the set $\mathbb{R} \times \{0\} \cup \{(0,1)\}$ and define a topology on X by taking as basic sets the intervals $]a,b[\times\{0\}$ on the x-axis and the basic neighborhoods $)]-\varepsilon,0[\times\{0\}) \cup \{(0,1)\} \cup (]0,\varepsilon[\times\{0\})$ of $(0,1)$, $\varepsilon > 0$. This defines T_1-manifold which is not Hausdorf and not regular. The sets $[\varepsilon,\varepsilon] \times \{0\}$ are compact nonclosed neighborhoods of $(0,0)$; the closure also contains $(0,1)$. Any homeomorphism of X either fixes each of the two points $(0,0)$ $(0,1)$ or interchanges them. Thus X is not homogeneous. It is instructive to follow the proof of 1.6 and to locate the point where in breaks down in this example. □

Now we turn to the class of homogeneous topological spaces which is of particular interest to us. The core definition is as follows:

Definition 1.7. (i) A *topological group* G is a group endowed with a topology such that multiplication $(x,y) \mapsto xy\colon G \times G \to G$ and inversion are continuous.

(ii) If G is a topological group and X a topological space, then a *topological group action* of G on X is a continuous action $(g,x) \mapsto g{\cdot}x : G \times X \to X$. We also say, that G *acts topologically* on X. □

Since inversion $x \mapsto x^{-1}$ in a group is an involution (i.e. satisfies $(x^{-1})^{-1} = x$), in a topological group G, it is clearly a homeomorphism of G onto itself.

Exercise E1.3. (i) Show that *a group G with a topology is a topological group if and only if the following function is continuous:* $(x,y) \mapsto xy^{-1}\colon G \times G \to G$.

(ii) Show that *for any subgroup H of a topological group G, the group G acts topologically on the quotient space $G/H \overset{\text{def}}{=} \{gH : g \in G\}$ with the quotient topology under the well defined action $(g, g'H) \mapsto gg'H$.* (Recall the definition of the quotient topology on the quotient space X/R of a space modulo an equivalence relation R on X: Let $q\colon X \to X/R$ denote the quotient maps given by $q(x) = R(x)$. Then a set V of X/R is open if and only if q^{-1} is open in X.) Show that this action is transitive. Conclude that G/H *is a homogeneous space.*

(iii) Show that *the quotient space G/H is a Hausdorff space if and only if H is a closed subgroup.*

[Hint. G/H is Hausdorff iff two different cosets have disjoint saturated neighborhoods. (A subset $S \subseteq G$ is saturated w.r.t. H iff $SH = S$.) If G/H is a Hausdorff space, it is a T_1-space so $\{H\}$ is a closed subset of G/H and so H is a closed subset of G. We prove the converse. Since G acts transitively on G/H on the left, we may consider the two cosets H and gH for $g \notin H$ and we have to produce disjoint saturated open sets containing H and gH, respectively. Since H is closed, there is a neighborhood W of g with $H \cap W = \emptyset$. Then $H \cap WH = \emptyset$, since $h_1 = wh_2$ implies $h_1h_2^{-1} = w \in H \cap H$. As $x \mapsto xg$ is continuous we find an open neighborhood U of 1 such that $Ug \subseteq W$. Since $(x,y) \mapsto xy$ is continuous at $(1,1)$ there is an identity neighborhood V such that $V^{-1}V \subseteq U$. Then $V^{-1}VgH \subseteq WH$ and so $H \cap V^{-1}VgH = \emptyset$. Then $VH \cap VgH = \emptyset$. Both sets VH and VgH are saturated, open as the unions $\bigcap_{h\in H} Vh$ and $\bigcap_{h\in H} Vgh$, respectively, and the first one contains $H = 1H$ and the second $gH = 1gH$.] □

Proposition 1.8. (i) *The space underlying a topological group is homogeneous.*

(ii) *Every quotient space G/H of a topological group modulo a subgroup H is a homogeneous space.* □

Exercise E1.4. Show that neither the closed half line $[0,\infty[$ nor the compact unit interval $\mathbb{I} = [0,1]$ can be the underlying spaces of a topological group, or even the quotient space of a topological group modulo a subgroup.

Examples 1.9. (i) Every group is a topological group when equipped with the discrete topology.

(ii) Every group is a topological group when equipped with the indiscrete topology.

(iii) $\mathbb{R}$ is a topological group with respect to addition. Also, $\mathbb{R} \setminus \{0\}$ is a topological group with respect to multiplication.

(iv) More generally, the additive group of $\mathbb{R}^n$ is a commutative topological group.

(v) Also more generally: Let $\mathbb{K}$ denote one of the fields $\mathbb{R}$, $\mathbb{C}$ or the division ring $\mathbb{H}$ of quaternions with the absolute value $|\cdot|$ in each case. Let $\mathbb{S}^n$, $n = 0, 1, 3$ denote the set $\{x \in \mathbb{K} : |x| = 1\}$ and $\mathbb{R}^< = \{x \in \mathbb{R} : 0 < x\}$. Then $\mathbb{R}^<$ and $\mathbb{K} \setminus \{0\}$ are topological groups under multiplication. The function

$$x \mapsto \left(|x|, \frac{x}{|x|}\right) : \mathbb{K} \setminus \{0\} \to \mathbb{R}^< \times \mathbb{S}^n$$

is an isomorphism of groups and a homeomorphism of topological spaces.

(vi) The groups $\mathrm{GL}(n, \mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ of invertible real or complex matrices are topological groups. □

Proposition 1.10. (i) *If H is a subgroup of a topological group G, then H is a topological group in the induced topology.*

(ii) *If $\{G_j : j \in J\}$ is a family of topological groups, then $G \stackrel{\text{def}}{=} \prod_{j\in J} G_j$ is a topological group.*

(iii) *If N is a normal subgroup of a topological group G, then the quotient group G/N is a topological group with respect to the quotient topology.* □

Proposition 1.11. *The closure of a subgroup is a subgroup, the closure of a normal subgroup is a normal subgroup.* □

Morphisms of topological groups

Definition 1.12. A *morphism of topological groups* $f: G \to H$ is a continuous homomorphism between topological groups.

Proposition 1.13. (a) *A homomorphism of groups $f: G \to H$ between topological groups is a morphism if and only if it is continuous at* 1.

(b) *The following conditions are equivalent:*

(i) *f is open.*

(ii) *For each $U \in \mathfrak{U}(1)$ the image $f(U)$ has a nonempty interior.*

(iii) *There is a basis $\mathfrak{B}$ of identity neighborhoods U such that $f(U)$ has a nonempty interior.*

(iv) *there is a basis of identity neighborhoods U of G such that $f(U)$ is an identity neighborhood of H.*

(v) *For all $U \in \mathfrak{U}_G(1)$ we have $f(U) \in \mathfrak{U}_H(1)$.*

(c) *For any normal subgroup N of G the quotient morphism $q: G \to G/N$ is continuous and open.* □

[Hint for (iii)⇒(iv): Let $U_1 \in \mathfrak{U}_G(1)$. We must find a $U \in \mathfrak{U}_G(1)$ such that $U \subseteq U_1$ and $f(U) \in \mathfrak{U}_H(1)$. Firstly, find $V_1 \in \mathfrak{U}_G(1)$ such that $V_1 V_1^{-1} \subseteq U_1$. Then let $V_2 \in \mathfrak{U}_G(1)$, $V_2 \subseteq \operatorname{int} V_1$ be such that $\operatorname{int} f(V_2) \neq \emptyset$ by (iv).

Now we find a $v_2 \in V_2$ such that $f(v_2) \in \operatorname{int} f(V_2) \subseteq \operatorname{int} f(\operatorname{int} V_1)$. Finally set $U = \operatorname{int} V_1 v_2^{-1}$. Then $1 = v_2 v_2^{-1} \in \operatorname{int} V_1 v_2^{-1}$, and so U is an open neighborhood of 1. Moreover, $U \subseteq V_1 V_1^{-1} \subseteq U_1$ and $1 = f(v_2) f(v_2)^{-1} \in \big(\operatorname{int} f(\operatorname{int} V_1)\big) f(v_2)^{-1} = \operatorname{int}\big(f(\operatorname{int} V_1) f(v_2)^{-1}\big) = \operatorname{int} f\big((\operatorname{int} V_1) v_2^{-1}\big) = \operatorname{int} f(U)$.]

Recall that for a filter basis $\mathfrak{B}$ on a set $\langle \mathfrak{B} \rangle$ denotes the filter generated by $\mathfrak{B}$ (i.e. the set of all supersets of sets from $\mathfrak{B}$) andnote that f is continuous at 1 iff $\mathfrak{U}_H(1) \subseteq \langle f(\mathfrak{U}_G(1)) \rangle$; conversely, condition (b)(v) may be rewritten as $\langle f(\mathfrak{U}_G(1)) \rangle \subseteq \mathfrak{U}_H(1)$. Thus an algebraic homomorphism $f: G \to H$ between topological groups is an open morphism of topological groups iff

$$\langle f(\mathfrak{U}_G(1)) \rangle = \mathfrak{U}_H(1).$$

Proposition 1.14. (Canonical decomposition) *A morphism of topological groups $f: G \to H$ with kernel $N = \ker f$ decomposes canonically in the form*

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle \mathrm{j}} \\ G/N & \xrightarrow[\quad f' \quad]{} & f(G), \end{array}$$

where $q: G \to N$ is the quotient morphism given by $q(g) = gN$, $j: f(G) \to H$ is the inclusion morphism, and $f': G/N \to f(G)$ is the bijective morphism of toplogical groups given by $f'(gN) = f(g)$.

The morphism is open if and only if $f(G)$ is open in H and f' is an isomorphism of topological groups, i.e. is continuous and open. The morphism f' is an isomorphism of topological groups if and only if f is open onto its image, that is, if and only if $f(U)$ is open in $f(H)$ for each open subset U of G. □

Let $G = \mathbb{R}_d$ the additive group of real numbers with its discrete topology, let $H = \mathbb{R}$, the additive group of real numbers with the natural topology and let $f: G \to H$ be the identity map. Then f is a bijective morphism of topological groups which is not an isomorphism, and f' may be identified with f in a natural way.

Another interesting application of the canonical decomposition of a function arises from the (topological) action of a (topological) group G on a (topological) space.

If a group G acts on X and on Y, then a function $f: X \to Y$ is called *equivariant* or a *morphism of actions* if $(\forall g \in G, x \in X)\, f(g{\cdot}x) = g{\cdot}f(x)$. Recall that for any subgroup H of a group G there is a natural action of G on the quotient space $G/H = \{gH : g \in G\}$, namely, the one given by $g{\cdot}(xH) = gxH$.

Now let G be a topological group acting topologically on a space X, then for each $x \in X$ there is an equivariant continuous bijection $f_x: G/G_x \to Gx$ given unambiguously by $f_x(gG_x) = gx$ and the continuous function $g \mapsto gx: G \to Gx$ decomposes into the composition $\mathrm{incl} \circ f_x \circ q$ of the continuous open quotient map $q = (g \mapsto gG_x): G \to G/G_x$, the function f_x, and the inclusion map $Gx \to X$. We have a commutative diagram of equivariant functions

$$\begin{array}{ccc} G & \xrightarrow{\;g\mapsto gx\;} & X\\ {\scriptstyle q}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{incl}}\\ G/G_x & \xrightarrow[\;f_x\;]{} & Gx. \end{array}$$

If G/G_x happens to be compact Hausdorff, then f_x is an equivariant homeomorphism, that is, an isomorphism of actions. Thus the quotient space G/G_x models the orbit $G{\cdot}x$.

A noteworthy observation is the following (Frattini argument):

If H is a subgroup of G acting transitively on X and containing the isotropy subgroup G_x then $H = G$.

Exercise E1.5. Prove the following

Proposition. (a) *Let A and B be an abelian groups. Define* $\mathrm{Hom}(A, B)$ *to be the set of all homomorphisms $f: A \to B$. Then B^A is an abelian group under componentwise group operations and* $\mathrm{Hom}(A, B)$ *is a subgroup of B^A.*

(b) *If B is a Hausdorff topological abelian group then B^A is a Hausdorff topological group with respect to the product topology and* $\mathrm{Hom}(A, B)$ *is a closed subgroup of B^A.*

(c) *The character group of a discrete abelian group A is defined as*

$$\widehat{A} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(A, \mathbb{R}/\mathbb{Z}).$$

Then $\widehat{A}$ is a closed subgroup of $(\mathbb{R}/\mathbb{Z})^A$ and thus is a compact abelian Hausdorff topological group.

The filter of identity neighborhoods: First applications

As we shall see shortly, the filter $\mathfrak{U} = \mathfrak{U}(1)$ of all identity neighborhoods is a very useful tool in topological group theory. We shall begin to use it now.

Lemma 1.15. (i) (The First Closure Lemma) *Let A be a subset of a topological group. Then $\overline{A} = \bigcap_{U \in \mathfrak{U}} AU = \bigcap_{U \in \mathfrak{U}} \overline{AU}$.*

(ii) (The Second Closure Lemma) *If A is a closed and K a compact subset of a topological group, then AK is a closed subset.*

Proof . (i) If $U \in \mathfrak{U}$ and $x \in \overline{A}$, then xU^{-1} is a neighborhood of x, and thus there is an $a \in A \cap xU^{-1}$. Write $a = xu^{-1}$ for some $u \in U$. Accordingly, $x = au \in AU \subseteq \overline{AU}$.

Conversely, assume that $x \in \bigcap_{U\in\mathfrak{U}} \overline{AU}$. First we claim that $x \in \bigcap_{U\in\mathfrak{U}} AU$; indeed let $U \in \mathfrak{U}$. Find a $U' \in \mathfrak{U}$ such that $U'U' \subseteq U$ by the continuity of multiplication at $(1,1)$. Now $x \in \overline{AU'} \subseteq AU'U' \subseteq AU$ by the preceding paragraph, and thus the claim is proved. Now let V by a neighborhood of x. We claim that $A \cap V \neq \emptyset$, thus showing $x \in \overline{A}$. Now $U \stackrel{\rm def}{=} V^{-1}x \in \mathfrak{U}$ is an identity neighborhood, and thus $x \in AU$, say $x = au$ with $a \in A$ and $u \in U$. Then $a = xu^{-1} \in xU^{-1} = xx^{-1}V = V$ and so $a \in A \cap V$ as asserted.

(ii) Let $g \in \overline{AK}$. We must show $g \in AK$. Now $(\forall U \in \mathfrak{U}(1))\, gU \cap AK \neq \emptyset$, and we can express this as $K \cap A^{-1}gU \neq \emptyset$ Now $\mathcal{C} \stackrel{\rm def}{=} \{K \cap \overline{A^{-1}gU} : U \in \mathfrak{U}(1)\}$ is a filter basis of closed subsets of K, and since K is compact, $\bigcap \mathcal{C} \neq \emptyset$. [1] Let k be a point in this interection. Then $k \in \bigcap_{U\in\mathfrak{U}(1)} A^{-1}gU = \overline{A^{-1}g}$ by (i) above. But $x \mapsto xg$ is a autohomeomorphism of G, whence $\overline{A^{-1}g} = \overline{A^{-1}}g$. But $x \mapsto x^{-1} : G \to G$ is likewise an autohomeomorphism of G. Thus $\overline{A^{-1}} = \overline{A}^{-1} = A^{-1}$ as A is closed. Thus there is an $a \in A$ such that $k = a^{-1}g$ and hence $g = ak \in AK$. Therefore AK is closed. □

Notice we did not use any separation axioms for G nor did we use that K is closed (which it need not be in a non-Hausdorff space). The following is a degenerate example which one should nevertheless keep in mind: Let G be a nonsingleton group and equip it with the indiscrete topology. Then $K = \{1\}$ is a compact subset which is not closed. The only nonempty closed subset A of G is G. Then $KA = G$ is closed.

Corollary 1.16. (i) $\overline{\{1\}} = \bigcap \mathfrak{U}$, *and*

(ii) $\overline{\{1\}}$ is a closed normal subgroup contained in every open and in every closed set meeting $\overline{\{1\}}$.

(iii) $\{1\}$ (hence any singleton subset) is dense in G if an only if G has the trivial topology $\{\emptyset, G\}$.

(iv) For every compact subset K of G the set $\overline{\{1\}}K$ is the closure of K.

(v) For every identity neighborhood U in G there is a closed identity neighborhood C such that $C \subseteq U$. Every neighborhood filter $\mathfrak{U}(g)$ has a basis of closed neighborhoods of g.

Proof . (i) follows from the First Closure Lemma,

(ii) firstly, closures of normal subgroups are normal subgroups. Secondly, let U be open and $U \cap \overline{\{1\}} \neq \emptyset$. Then $1 \in U$. Thus $\overline{\{1\}} \subseteq U$ by part (i). If A is a closed subset and $A \cap N \neq \emptyset$, then $N \not\subseteq (G \setminus A)$ and thus $N \cap (G \setminus A) = \emptyset$ by the preceding, and thus $N \subseteq A$.

(iii) Let U be nonempty and open in G. If $\{1\}$ is dense, then $G = \overline{\{1\}} \subseteq U$ by (ii).

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[1] Recall that the Heine Borel covering property for open sets is equivalent to saying that each filter basis of closed sets has a nonempty intersection.

(iv) The set $\overline{\{1\}}K$ is closed by 1.15(ii), and thus $\overline{K} \subseteq \overline{\{1\}}K$. But $\overline{\{1\}}K \subseteq \overline{\{1\}K} = \overline{K}$ by the continuity of the multiplication. Thus $\overline{\{1\}}K = \overline{K}$.

(v) If $U \in \mathfrak{U}$, then by the continuity of multiplication there is a $V \in \mathfrak{U}$ such that $VV \subseteq U$. Set $C = \overline{V}$. By the First Closure Lemma, $C = \overline{V} \subseteq VV \subseteq U$. Thus $\mathfrak{U}(1)$ has a basis of closed sets, and since G is homogeneous, every neighborhoodfilter has a basis of closed neighborhoods.

□

Separation Axioms in topological groups

Theorem 1.17. *In a topological group G, every neighborhood filter of a point has a basis of closed neighborhoods, and the following conditions are equivalent:*

(i) *G is a T_0–space.*
(ii) *$\{1\}$ is closed.*
(iii) *G is a T_1–space.*
(iv) *G is a regular Hausdorff space, i.e. a T_3–space.*

Proof . A T_0–space in which every point has a neighborhood basis of closed neighborhoods is a T_3–space (see Lecture Notes "Introduction to Topology", 1.38). Thus (i) implies (iv), and $(T_3)\Rightarrow(T_2)\Rightarrow(T_1)$. □

There are also pedestrian proofs of the individual implications: (i)⇒(ii): Let $x \neq 1$. By (i) there is an open set U containing exactly one of 1 or x. if $1 \in U$ then $x \notin U$. Now $1 \in U^{-1}$ and thus $x \in U^{-1}x$; thus $U^{-1}x$ is an open neighborhood of x which does not contain 1, for if it did, then $1 = u^{-1}x$ for some $u \in U$, and then $x = u \in U$. Thus every element $x \neq 1$ has an open neighborhood missing 1, and thus (ii) is proved.

(ii)⇒(iii): This follows from the homogeneity of G.

(iii)⇒(iv): Assume $x \neq y$ in G. Then $1 \neq xy^{-1}$. By (iii), $G \setminus \{xy^{-1}\} \in \mathfrak{U}$, and by continuity of $(g, h) \mapsto g^{-1}h$ there is a $V \in \mathfrak{U}$ such that $V^{-1}V \subseteq G \setminus \{xy^{-1}\}$. If $g = Vx \cap Vy$, then $g = vx = wy$ with $v, w \in V$, whence $xy^{-1} = v^{-1}w \in V^{-1}V \subseteq G \setminus \{xy^{-1}\}$, a contradiction. Thus Vx and Vy are two disjoint neighborhoods of x and y, respectively.

(iv)⇒(i): Trivial.

Corollary 1.18. *A quotient group of a topological group G modulo a normal subgroup N is a Hausdorff group if and only if N is closed.* □

(See also Exercise E1.3.)

Corollary 1.19. *For every topological group G, the factor group $G/\overline{\{1\}}$ is a regular Hausdorff group and for each continuous homomorphism $f\colon G \to H$ into* *a Hausdorff group there is a unique morphism* $f'\colon G/\overline{\{1\}} \to H$ *such that* $f = f'q$ *with the quotient morphism* $q\colon G \to G/\overline{\{1\}}$. □

Proposition 1.20. *Let* G *be a topological group, and* U *an open subset. Set* $N = \overline{\{1\}}$. *Then* $UN = U$. *Every open set is the union of its* N*-cosets.*

Proof. By 1.16, N is contained in every open set U meeting N. Let $x \in UN$. Then $x = un$ with $u \in U$ and $n \in N$. Then $n = u^{-1}x \in U^{-1}x$. Thus $1 \in N \subseteq U^{-1}x$ and therefore $1 \in U^{-1}x$, i.e., $x \in U$. □

Corollary 1.21. *Let* $\big(G, \mathfrak{O}(G)\big)$ *be any topological group and let* $q\colon G \to G/\overline{\{1\}}$ *be the quotient morphism of* G *onto the Hausdorff topological group*

$$\big(G/\overline{\{1\}}, \mathfrak{O}(G/\overline{\{1\}})\big).$$

Then $U \mapsto q^{-1}(U)\colon \mathfrak{O}(G/\overline{\{1\}}) \to \mathfrak{O}(G)$ *is a* $\bigcap - \bigcup$*-preserving bijection.* □

There is an obvious converse to all of this:

Assume that $f\colon G \to T$ *is a surjective morphisms of groups and that* T *is a Hausdorff topological group. Then* $\{f^{-1}(U) : U$ *is open in* $T\}$ *is a group topology on* G *such that* $\overline{\{1\}} = \ker f$.

All group topologies on G arise in this fashion from a group homomorphism into a Hausdorff group such that $\ker f = \overline{\{1\}}$.

The Identity Component

Definition 1.22. For a topological group G let G_0 denote the connected component of the identity, short the *identity component.* Similarly let G_a denote the arc component of the identity, the *identity arc component.* □

Definition 1.23. A subgroup H of a topological group G is called *characteristic* if it is invariant under all automorphisms of G, i.e., all continuous and continuously inversible group homomorphisms. It is called *fully characteristic* if it is invariant under all (continuous!) endomorphisms. □

Every fully characteristic subgroup is characteristic. The inner automorphisms $x \mapsto gxg^{-1}\colon G \to G$ are continuous and continuously invertible. Hence every characteristic subgroup is invariant under all inner automorphisms, i.e. is normal. The subgroup $\overline{\{1\}}$ is an example of a fully characteristic subgroup.

Proposition 1.24. *The identity component* G_0 *and the identity arc component* G_a *of any topological group* G *are fully characteristic subgroups of* G. *The identity component* G_0 *is closed. The factor group* G/G_0 *is a totally disconnected Hausdorff topological group—irrespective of whether* G *itself is Hausdorff or not.* □

Exercise E1.6.A. Prove the following assertions.
(i) *A subgroup H of a topological group is open iff it contains a nonempty open subset.*

(ii) *If H is an open subgroup of a topological group G, then H is closed and $G_0 \subseteq H$. The quotient space $G/H = \{gH : g \in G\}$ is discrete.*
(iii) *If G contains a connected subset with nonempty interior, then G_0 is open and G/G_0 is discrete.*
(iv) *If G is a locally connected topological group and $f: G \to H$ is an open morphism of topological groups, then the identity component G_0 of G is mapped onto the identity component of H.*

Notice that the hypothesis of (iii) is satisfied if G is locally connected iff $\mathfrak{U}(1)$ has a basis of connected neighborhoods.

Proposition 1.25. *Let G be a topological group and let $\mathcal{OC}(G)$ be the set of open and closed subsets of G. Then the following conclusions hold:*

(i) $U \in \mathcal{OC}(G)$ implies $UG_0 = G_0U = U$.
(ii) Let $q: G \to G/G_0$ denote the quotient homomorphism. Then $U \mapsto q^{-1}U : \mathcal{OC}(G/G_0) \to \mathcal{OC}(G)$ is a bijection.

Proof. (i) If $U = \emptyset$, nothing is to be proved. If $u \in U$ then uG_0 is a connected set and $U \cap uG_0$ is nonempty open closed in uG_0. Then $uG_0 \subseteq U$..

(ii) In view of (i) we note that $V \mapsto q(V) : \mathcal{OC}(G) \to \mathcal{\prime C}G/G_0$ is a well-defined function inverting $U \mapsto q^{-1}(U)$. □

This allows us to give an alternative proof of the assertion that G/G_0 is totally disconnected. (See 1.24.) Let $q: G \to G/G_0$ be the quotient morphism $H = q^{-1}\big((G/G_0)_0\big)$. Show $H = G_0$. Consider $q|H : H \to (G/G_0)_0$. Without loss of generality assume that G/G_0 is connected; show G is connected. Now $\mathcal{OC}(G/G_0) = \{\emptyset, G/G_0\}$. Then by (ii), $\mathcal{OC}(G) = \{\emptyset, G\}$. □

We notice that Proposition 1.25 is just a special case of a more general one which has essentially the same proof:

Proposition 1.26. *Let X be a topological space and let $\mathcal{OC}(X)$ be the set of open and closed subsets of X. Let R denote the equivalence relation of connectivity on X. Then the following conclusions hold:*

(i) $U \in \mathcal{OC}(X)$ implies $U = \bigcup_{u \in U} R(u)$.
(ii) Let $q: X \to X/R$ denote the quotient map. Then $U \mapsto q^{-1}U : \mathcal{OC}(X/R) \to \mathcal{OC}(X)$ is a bijection.
(iii) X/R is a totally disconnected T_1 space. □

Exercise E1.8. (a) Prove the following

Lemma. *Assume that G is a topological group and $\mathcal{OC}(G)$ is the set of all open closed subsets. Then $G_1 \stackrel{\rm def}{=} \bigcap\{U \in \mathcal{OC}(G) : 1 \in U\}$ is a characteristic subgroup.*
[Hint. Each homeomorphism of G permutes $\mathcal{OC}$. Hence G_1 is invariant under all homeomorphisms of G fixing 1, in particular under $x \mapsto x^{-1}$ and all automorphisms. For a proof of $G_1G_1 \subseteq G_1$, let $g \in G_1$; we must show $gG_1 \subseteq G_1$. It suffices to show $gG_1 \subseteq U$, that is, $G_1 \subseteq g^{-1}U$ for any $U \in \mathcal{OC}(G)$ containing 1. Argue that it suffices to show that $g^{-1}U$ is open-closed and contains 1. □

This requires no compactness nor separation.

One can define, by *transfinite induction*, for each ordinal α a characteristic subgroup G_α as follows:

Assume that G_α has been defined for all ordinals $\alpha < \beta$. Then we set

$$G_\beta = \begin{cases} (G_\alpha)_1 & \text{if } \beta = \alpha + 1, \\ \bigcap_{\alpha<\beta} G_\alpha & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

(b) Show that $G_0 \subseteq G_\alpha$ for all α.

[Hint. Show that $G_0 \subseteq G_1$ and then use transfinite induction.]

(c) Show the following

Proposition. *There is an ordinal γ such that $G_\gamma = G_0$.*

[Hint. For cardinality reasons, there is an ordinal γ such that $(G_\gamma)_1 = G_\gamma$. This means that the only nonempty open closed subset of G_γ is G_γ. That is, G_γ is connected and thus $G_\gamma \subseteq G_0$.] □

A group which is connected but not arcwise connected

Exercise 1.6.B. (i) Assume that we have a sequence $\varphi_n\colon G_{n+1} \to G_n$, $n \in \mathbb{N}$ of morphisms of compact groups:

$$G_1 \stackrel{\varphi_1}{\leftarrow} G_2 \stackrel{\varphi_2}{\leftarrow} G_3 \stackrel{\varphi_3}{\leftarrow} G_4 \stackrel{\varphi_4}{\leftarrow} \cdots$$

Then the *limit* of this sequence as $G = \lim_{n\in\mathbb{N}} G_n \subseteq \prod_{n\in\mathbb{N}} G_n$ is simply given by $\{(g_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} G_n \mid (\forall n \in \mathbb{N})\, \varphi_n(g_{n+1}) = g_n\}$.

Then G is a compact topological group.

(ii) Choose a natural number p and keep it fixed, for instance $p = 2$.

Set $G_n = \mathbb{T}$ for all $n \in \mathbb{N}$ and define $\varphi_n(g) = p{\cdot}g$ for all $n \in \mathbb{N}$ and $g \in \mathbb{T}$. (It is customary, however, to write p in place of φ_p):

$$\mathbb{T} \stackrel{p}{\leftarrow} \mathbb{T} \stackrel{p}{\leftarrow} \mathbb{T} \stackrel{p}{\leftarrow} \mathbb{T} \stackrel{p}{\leftarrow} \cdots$$

The projective limit of this system is called the *p-adic solenoid* $\mathbb{T}_p$. □

If $p = 2$ then $\mathbb{T}_2$ is also called the *dyadic solenoid.*

(iii) Set $G_n = \mathbb{Z}(p^n) = \mathbb{Z}/p^n\mathbb{Z}$. Define $\varphi_n\colon \mathbb{Z}(p^{n+1}) \to \mathbb{Z}(p^n)$ by $\varphi_n(z + p^{n+1}\mathbb{Z}) = z + p^n\mathbb{Z}$:

$$\mathbb{Z}(p) \xleftarrow{\varphi_1} \mathbb{Z}(p^2) \xleftarrow{\varphi_2} \mathbb{Z}(p^3) \xleftarrow{\varphi_3} \mathbb{Z}(p^4) \xleftarrow{\varphi_4} \cdots$$

The projective limit of this system is called the *group* $\mathbb{Z}_p$ *of p-adic integers.*

Let us discuss these examples in the following exercises:

(iv) Observe that the *bonding maps* φ_1, φ_2, ... are morphisms of rings. Prove that $\mathbb{Z}_p$ is a compact ring with continuous multiplication so that all limit maps $f_n\colon \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ are morphisms of rings.

(v) Define $\eta\colon \mathbb{Z} \to \mathbb{Z}_p$ by $\eta(z) = (z + p^n\mathbb{Z})_{n\in\mathbb{N}}$. Show that this is a well defined injective morphism of rings.

(vi) Prove the following statement: For an arbitrary element

$$g = (z_n + p^n\mathbb{Z})_{n\in\mathbb{N}} \in \mathbb{Z}_p,$$

the sequence $\big(\eta(z_n)\big)_{n\in\mathbb{N}}$ converges to g in $\mathbb{Z}_p$. Conclude that η has a dense image.

(vii) Show that $\mathbb{Z}_p$ is totally disconnected.

(viii) Show that the *limit map* $f_m\colon \mathbb{Z}_p \to \mathbb{Z}/p^m\mathbb{Z}$ has kernel $\{(z_n + p^n\mathbb{Z})_{n\in\mathbb{N}} \mid z_m \equiv 0 \pmod{p^m}\}$. Show that it is $p^m\mathbb{Z}_p = \overline{\eta(p^m\mathbb{Z})}$. Prove that the subgroups $p^m\mathbb{Z}_p$ are open and closed and form a basis for the filter of neighborhoods of 0.

(ix) Show that the limit of the system

$$(\frac{1}{p}{\cdot}\mathbb{Z}/\mathbb{Z}) \xleftarrow{p} (\frac{1}{p^2}{\cdot}\mathbb{Z}/\mathbb{Z}) \xleftarrow{p} (\frac{1}{p^3}{\cdot}\mathbb{Z}/\mathbb{Z}) \xleftarrow{p} (\frac{1}{p^4}\mathbb{Z}/\mathbb{Z}) \xleftarrow{p} \cdots$$

is a group $\mathbb{Z}'_p$ isomorphic to $\mathbb{Z}_p$, and that $\mathbb{Z}'_p \subseteq \mathbb{T}_p$. Show that $\mathbb{Z}'_p$ is the kernel of the map $\varphi\colon \mathbb{T}_p \to \mathbb{T}$, $\varphi\big((r_n + \mathbb{Z})_{n\in\mathbb{N}}\big) = pr_1 + \mathbb{Z}$. [Note that $(r_n + \mathbb{Z})_{n\in\mathbb{N}} \in \ker\varphi$ iff $r_1 + \mathbb{Z} \in \frac{1}{p}\mathbb{Z}/Z, r_2 + \mathbb{Z} \in \frac{1}{p^2}\mathbb{Z}/Z, \ldots$ and this means $(r_n + \mathbb{Z})_{n\in\mathbb{N}} \in \mathbb{Z}'_p$.]

(x) Show that $\mathbb{Z}_p$ is torsion-free; that is, it has no elements of finite order. [Remark that for any element $y \in \mathbb{Z}/p^{n+m}\mathbb{Z}$ if $x = \varphi_n(y) \in \mathbb{Z}/p^n\mathbb{Z}$ satisfies $k{\cdot}x = 0$ for a smallest natural number k, then $kp^m{\cdot}y = 0$ and $kp^m|k'$ whenever $k'{\cdot}y = 0$.]

(xi) Define $\pi\colon \mathbb{Z}'_p \times \mathbb{R} \to \mathbb{T}_p$ by $\pi\big((z_n + \mathbb{Z})_{n\in\mathbb{N}}, r)\big) = (z_n + p^{-n}r + \mathbb{Z})_{n\in\mathbb{N}}$. Show that π is surjective and that the kernel of π is $\{\big((p^{-n}z + \mathbb{Z})_{n\in\mathbb{N}}, -z\big)$ and that this subgroup is isomorphic to $\mathbb{Z}$. Show that $\mathbb{Z}'_p \times \mathbb{R}$ has an open zero-neighborhood $U = \mathbb{Z}'_p \times]-\frac{1}{4}, \frac{1}{4}[$ which is mapped homeomorphically onto an open zero neighborhood $\pi(U) = \varphi_1^{-1}(]-1/4, 1/4[)$ of $\mathbb{T}_p$, where $\varphi_1 : \mathbb{T}_p \to \mathbb{T}$ is the projection onto the first component. So $\mathbb{Z}'_p \times \mathbb{R}$ and $\mathbb{Z}_p$ are locally isomorphic. The identity component of $\mathbb{Z}'_p \times \mathbb{R}$ is $\{0\} \times \mathbb{R}$. Its image under π is not equal to the identity component of $\mathbb{T}_p$ which is $\mathbb{T}_p$ itself.

The arc component $(\mathbb{T}_p)_a$ is $\pi(\{0\} \times \mathbb{R})$.

The group $\mathbb{T}_p$ is compact, connected, commutative; it has uncountably many arc components, that is, cosets modulo $(\mathbb{T}_p)_a$. □

Invariant neighborhoods

The following is a slight generalisation of the Second Closure Lemma 1.15(ii).

Lemma 1.27. *Assume that G is a topological group acting (continuously) on a topological space X. Let K be a compact subset of G and A a closed subset of X. Then $K{\cdot}A$ is closed in X.*

Proof . Let $y \in \overline{K{\cdot}A}$. Then for every $U \in \mathfrak{U}(y)$ we have $U \cap K{\cdot}A \neq \emptyset$, and thus $K_U = \{g \in K : (\exists a \in A)\, g{\cdot}a \in U\} \neq \emptyset$. The collection $\{K_U : U \in \mathfrak{U}(y)\}$ is a filter basis on the compact space K and thus we find a $h \in \bigcap_{U \in \mathfrak{U}(y)} K \cap \overline{K_U}$. Thus for any $j \stackrel{\text{def}}{=} (U, V) \in \mathfrak{U}(y) \times \mathfrak{U}(h)$ the set $F_j \stackrel{\text{def}}{=} K_U \cap V$ is not empty and contained in V. For $g \in F_j$ there is a $b \in A$ such that $g{\cdot}b \in U$. Thus $b \in g^{-1}{\cdot}U$. Hence the set of all $A \cap F_j^{-1}{\cdot}U \subseteq A \cap V^{-1}{\cdot}U$, as $j = (U, V)$ ranges through $\mathfrak{U}(y) \times \mathfrak{U}(h)$ is a filter basis converging to $a \stackrel{\text{def}}{=} h^{-1}{\cdot}y$ by the continuity of the action. Since A is closed we have $a \in A$. Thus $y = h{\cdot}a \in K{\cdot}A$. □

Exercise E1.7. Derive the Second Closure Lemma 1.15(ii) from Lemma 1.26. If K is a compact subset and A is a closed subset of a topological group, then KA and AK are closed subsets. □

Lemma 1.28. *Assume that G is a topological group acting (continuously) on a topological space X such that $G{\cdot}x = \{x\}$. Then for any open set U containing x and every compact subset K of G, the set $V \stackrel{\text{def}}{=} \bigcap_{g \in K} g{\cdot}U$ is open. Thus if G itself is a compact group, then x has arbitrarily small invariant neighborhoods.*

Proof . Let $A \stackrel{\text{def}}{=} X \setminus U$. Then A is closed and $K{\cdot}A = \bigcup_{g \in K} g{\cdot}A = \bigcup_{g \in K} g{\cdot}(X \setminus U) = \bigcup_{g \in K}(X \setminus g{\cdot}U) = X \setminus \bigcap_{g \in K} g{\cdot}U = X \setminus V$. Since $K{\cdot}A$ is closed by 1.26, its complement V is open. □

An alternative proof exists which does not use 1.26 but

Wallace's Lemma. *Let A and B be compact subspaces of spaces X and Y, respectively, and assume that U is an open subset of $X \times Y$ containing $A \times B$. Then* *there are open subsets V and W of X and Y containing A and B, respectively, such that $V \times W \subseteq U$.*

For a proof see e.g. Introduction to Topology, Summer 2005, Lemma 3.21. The proof uses only the definitions of compactness and the product topology of a product space of two factors.

Now for a proof of Lemma 1.28. Let $\alpha\colon G \times X \to X$ be the continuous function given by $\alpha(g, y) = g^{-1}{\cdot}y$. Then $\alpha^{-1}(U)$ is an open neighborhood of $K \times \{x\} \subseteq G \times X$. Then by Wallace's Lemma there are open neighborhoods V of K in G and W of x in X such that $V^{-1}{\cdot}W = \alpha(V \times W) \subseteq U$, and thus $W \subseteq \bigcap_{g \in V} g{\cdot}U \subseteq \bigcap_{g \in K} g{\cdot}U$. □

Actually, this proof shows even a bit more: The neighborhood V of K can replace K!

Yet another proof is given on page 9 of K. H. Hofmann and S. A. Morris, The Structure of Compact Groups, De Gruyter, Berlin, 1998.

Now, proceeding towards the next theorem let us observe the following: If U is a compact open neighborhood of the identity in a topological group G, and if $A = G \setminus U$, then there is a $V \in \mathfrak{U}(1)$ such that $U \cap AV = \emptyset$. For if this fails, then the filter basis of all $\overline{U \cap AV}$, $V \in \mathfrak{U}(1)$ has an element $u \in U$ in its intersection. Then $u \in \overline{U} \cap \bigcap_{V \in \mathfrak{U}(1)} \overline{AV}$. By the First Closure Lemma 1.15(i),

$$\bigcap_{V \in \mathfrak{U}(1)} \overline{AV} = \overline{A}.$$

Since U is clopwn, both U and its complement A are closed. So $u \in \overline{U} \cap \overline{A} = U \cap A = \emptyset$, a contraduction.

Proposition 1.29. *Assume that U is a compact open identity neighborhood in a topological group. Then there is a compact open subgroup H contained in U, in fact $UH = U$.*

Proof . Again set $A = G \setminus U$. Find a symmetric identity neighborhood $V = V^{-1}$ such that $U \cap AV = \emptyset$. Then $UV \cap A = \emptyset$, i.e. $UV \subseteq U$. By induction, $UV^n \subseteq U$ where $V^n = \underbrace{V \cdots V}_{n \text{ times}}$. Set $H = \bigcup_{n=1}^{\infty} V^n$. Then H is an open subgroup and $UH = U$. □

Since we can also choose V so that $VUV = U$, we may assume that $H \subseteq HU = UH = U$.

In the following we invoke a theorem on connectivity in compact spaces. We first present the theorem and its proof for compact metric spaces; then we formulate the general version and its proof. This may facilitate the understanding of the proof in the general case.

Theorem C metric. *Let (X, d) by a compact metric space. Then every component has a neighborhood basis of clopen subsets.*

Proof. For each $\varepsilon > 0$ we define R_ε to be the set of all pairs (x, y) such that there is a finite sequence $x_0 = x, x_1, \dots, x_n = y$ such that $d(x_{j-1}, x_j) < \varepsilon$; we shall call such a sequence an ε–chain.

Then R_ε is reflexive, symmetric, and transitive. Hence R_ε is an equivalence relation. Write $U_\varepsilon(x) = \{u \in X : d(x, u) < \varepsilon\}$. Then for each x' in the equivalence class $R_\varepsilon(x)$ of x, the set $U_\varepsilon(x')$ is a neighborhood of x' which is contained in $R_\varepsilon(x)$. Hence the relation R_ε is open and therefore closed as the complement of all other equivalence classes. Let S be the intersection of the clopen equivalence relations R_ε as ε ranges through the positive real numbers. Clearly S is an equivalence relation and is closed in $X \times X$. Then every pair of elements in S is R_ε-equivalent for all $\varepsilon > 0$. Set $C = S(x)$ and let R denote the equivalence relation of connectivity. The component $R(x)$ of x is contained in C. We aim to show that C is connected. Then $C = R(x)$ for all $x \in C$. Thus $S = R$. So $R(x) = \bigcap_{\varepsilon>0} R_\varepsilon(x)$ and then, by the Filter Basis Lemma, the sets $R_\varepsilon(x)$ form a basis of the neighborhoods of $C = R(x)$. This will complete the proof.

Now suppose that C is not connected. Then $C = C_1 \dot\cup C_2$ with the disjoint nonempty closed subsets of C. Then by the compactness of the space $C_1 \times C_2$, the continuous function $d|(C_1 \times C_2)$ having values in $]0, \infty[$ has a positive minimum $\varepsilon > 0$.

Now let $U_{\varepsilon/3}(C_j) = \{x \in X : (\exists c \in C_j)\ d(c, x) < \varepsilon/3\}$, $j = 1, 2$, and set $D = X \setminus \big(U_{\varepsilon/3}(C_1) \cup U_{\varepsilon/3}(C_2)\big)$. Let $0 < r < \varepsilon/3$. If $x \in C_1$ and $c_2 \in C_2$, then $(x, c_2) \in R_r$ since $C \in R_r(x)$. Now consider an r–chain $x = x_0, x_1, \dots, x_n = c_2$: There is a smallest $k \in \{1, \dots, n\}$ such that $x_k \notin U_{\varepsilon/3}(C_1)$. There is a $c_1 \in C_1$ such that $d(c_1, x_{k-1}) < \varepsilon/3$. We claim that $x_k \notin U_{\varepsilon/3}(C_2)$. Indeed suppose that $x_k \in U_{\varepsilon/3}(C_2)$. Then there is a $c_2 \in C_2$ such that $d(x_k, c_2) < \varepsilon/3$. Then $d(c_1, x_{k-1}) < \varepsilon/3$, $d(x_{k-1}, x_k) < \varepsilon/3$, $d(x_k, c_2) < \varepsilon/3$ and thus $d(c_1, c_2) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon = \min d(C_1 \times C_2)$, a contradiction. Thus there is a k such that $x_k \in D$. Thus $R_r(x) \cap D \neq \emptyset$. So the sets $R_r(x) \cap D$ form a filter basis on the compact space D. Let y be in its intersection. Then $y \in \bigcap_{r>0} R_r(x) = C$ and $y \in D$, whence $d \in C \cap D = \emptyset$: a contradiction. This shows that C is connected as asserted and completes the proof. □

Theorem C. *Let X by a compact Hausdorff space. Then every component has a neighborhood basis of clopen subsets.*

Proof. Let U be a neighborhood of the diagonal Δ in $X \times X$. By replacing U by $\{(u, v) : (u, v), (v, u) \in U\}$ if necessary, we may assume that U is symmetric. We define R_U to be the set of all pairs (x, y) such that there is a finite sequence $x_0 = x, x_1, \dots, x_n = y$ such that $(x_{j-1}, x_j) \in U$; we shall call such a sequence a U–chain. Then R_U is reflexive, symmetric, and transitive. Hence R_U is an equivalence relation. Write $U(x) = \{u \in X : (x, u) \in U\}$. Then $U(x)$ is a neighborhood of x. Since $U(x') \subseteq R_U(x)$ for each $x' \in R_U(x)$, the relation R_U is open and therefore closed as the complement of all other equivalence classes. Let S be the intersection of the clopen equivalence relations R_U as U ranges through the filter basis $\mathfrak{U}_s(\Delta)$ of symmetric neighborhoods of Δ. Then S is an equivalence relation and S is closed in $X \times X$. Then every pair of elements in C is R_U-equivalent for all $U \in \mathfrak{U}_s(\Delta)$. Let R denote the connectivity relation on X. Set $C = S(x)$. The component $R(x)$ of x is contained in C. We aim to show that C is connected. Then $C = R(x)$ for all x and thus $R = S$. So $R(x) = \bigcap_{U \in \mathfrak{U}_s(\Delta)} R_U(x)$, and then, by the Filter Basis Lemma, the sets $R_U(x)$ form a basis of the neighborhoods of $C = R(x)$. This will complete the proof.

Now suppose that C is not connected. Then $C = C_1 \dot{\cup} C_2$ with the disjoint nonempty closed subsets of C. We claim that there is an open symmetric neighborhood $U \in \mathfrak{U}(\Delta)$ of the diagonal Δ in $X \times X$ such that the set $U(C_1) \cap C_2$ is empty. [It suffices to show that every open neighborhood W of a compact subset K of X contains one of the form $U(K)$. Proof by contradiction: If not, then for all open neighborhoods U of the diagonal in $X \times X$, $U(K) \cap (X \setminus W)$ is not empty and the collection of sets $U(K) \cap (X \setminus W)$ is a filter basis on the compact space $X \setminus W$. Let z be in the intersection of the closures of the sets in this filterbasis. Since X is Hausdorff, the diagonal is closed in $X \times X$ and by the Normality Lemma is the intersection of its closed neighborhoods. Thus z in the intersection of all $U(K)$ for all closed U and this is K. Thus $z \in K \setminus W = \emptyset$, a contradiction!]

Recall that for two subsets $A, B \subseteq X \times X$ we set $A \circ B = \{x, z) \in X \times X : (\exists y \in X)\ (x, y) \in A \text{ and } (y, z) \in B\}$. Now assume that W is an open neighborhood of the diagonal such that $W \circ W \circ W \subseteq U$ and set $D = X \setminus \big(W(C_1) \cup W(C_2)\big)$. Now let $V \in \mathfrak{U}(\Delta)$, $V \subseteq W$. By replacing V by $\{(u, v) : (u, v), (v, u) \in V\}$ if necessary, we may assume that V is symmetric.

If $x \in C_1$ and $c_2 \in C_2$, then $(x, c_2) \in R_V$ since $C \in R_V(x)$. Now any V–chain $x = x_0, x_1, \ldots, x_n = c_2$ has at least one element in D. Thus $R_V(x) \cap D \neq \emptyset$ and so the sets $R_V(x) \cap D$ form a filter basis on the compact space D. Let y be in its intersection. Then $y \in \bigcap_{V \in \mathfrak{U}_s(\Delta)} R_V(x) = C$ and $y \in D$, whence $y \in C \cap D = \emptyset$: a contradiction. This shows that C is connected as asserted and completes the proof. □

Example C. There is a locally compact space X containing a point x whose connected component $R(x)$ which is not the intersection C of all open closed neighborhoods of x.

Proof . Consider the compact space $Y \stackrel{\text{def}}{=} [-1, 1] \times (\{0\} \cup \{1/n : n \in \mathbb{N}\})$. Set $X = Y \setminus \{(0, 0)\}$. Then X is a locally compact space in which the component $R(x)$ of $x = (1, 0)$ is $]0, 1] \times \{0\}$ while the intersection C of all open closed neighborhoods of $(1, 0)$ is $([-1, 1] \setminus \{(0, 0)\}) \times \{0\}$. □

The example shows in particular a locally compact space in which not every component has a basis of clopen neighborhoods.

Theorem 1.30. *Let G be a locally compact totally disconnected group. Then for any identity neighborhood U there is a compact open subgroup H contained in U. If G is compact there is a compact open normal subgroup N with $N \subseteq U$.*

Proof . Let K be a compact neighborhood of 1. Then K is totally disconnected because any component of K is a connected subset of G. By Theorem C, the filter $\mathfrak{U}(1)$ of identity neighborhoods has a basis of clopen neighborhoods U of K, and since K contains an open subset W of G all sufficiently small clopen subsets U are contained in W and thus are open in G and closed, hence compact in K. Thus they are compact hence closed in G (since G is Hausdorff). By 1.29, every such clopen identity neighborhood U contains an open subgroup H such that $UH = U$. If G is compact then by 1.28, $N \stackrel{\text{def}}{=} \bigcap_{g\in G} gHg^{-1}$ is open. Also, N is invariant under all inner automorphisms. □

One also expresses this fact by saying that a locally compact totally disconnected group G has arbitrarily *small compact open subgroups* and that a totally disconnected compact group has *arbitrarily small compact open normal subgroups* N. For each of these, the factor group is finite and discrete. Thus we might say that G is approximated by the finite subgroups G/N. Therefore compact totally disconnected groups are also called *profinite groups.* They occur in the Galois theory of infinite field extensions.

Exercise E1.9. (i) Let $\{G_j : j \in J\}$ be a family of finite groups and form the totally disconnected compact group $G = \prod_{j\in J} G_j$. Identify a neighborhood basis of 1 consisting of open normal subgroups.

(ii) Let G be a compact totally disconnected group and $\mathcal{N}$ the set of open normal subgroups. Then the function $f\colon G \to \prod_{N\in\mathcal{N}} G/N$, $f(g) = (gN)_{N\in\mathcal{N}}$ is an injective morphism and homeomorphism onto its image.

(iii) Prove the following conclusion:

Theorem. *For a topological group G the following statements are equivalent.*

(1) *G is isomorphic to a closed subgroup of a product of finite groups.*

(ii) *G is compact totally disconnected.* □

We resume an example which we discussed in Exercise 1.6.B.

Example 1.31. Let p be a natural number, $p \geq 2$, for instance a prime number. In the compact totally disconnected group $P \stackrel{\text{def}}{=} \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ consider the closed subgroup $\mathbb{Z}_p$ of all N-tuples $(z_n + p^n\mathbb{Z})_{n\in\mathbb{N}}$ such that $z_{n+1} - z_n \in p^n\mathbb{Z}$.

Then $\mathbb{Z}_p$ is a compact totally disconnected abelian group with a basis of identity neighborhoods $\{p^n\mathbb{Z}_p : n \in \mathbb{N}\}$. The subgroup of all $(z + p^n)_{n\in\mathbb{N}}$, $z \in \mathbb{Z}$ is algebraically isomorphic to $\mathbb{Z}$ and is dense in $\mathbb{Z}_p$. Thus $\mathbb{Z}_p$ is a "compactification" of $\mathbb{Z}$. Elements are close to zero if they are divisible by large powers of p.

The group $\mathbb{Z}_p$ is called the *group of p-adic integers.*

The additive group P is a ring under componentwise multiplication. The subgroup $\mathbb{Z}_p$ is closed under multiplication. Thus $\mathbb{Z}_p$ is in fact a compact ring with a continuous multiplication, containing $\mathbb{Z}$ as a dense subring.

The underlying topological space of $\mathbb{Z}_p$ is homeomorphic to the Cantor set.

There is an interesting application of connectivity.

Theorem 1.32. *Let G be a connected topological group and N a totally disconnected normal subgroup. Then N is central, that is,*

$$(\forall g \in G,\, n \in N)\, gn = ng.$$

Proof . Let $n \in N$. The continuous function $g \mapsto gng^{-1}n^{-1}: G \to N$ maps a connected space into a totally disconnected space, and the image contains 1. Then this function is constant and takes the value 1. □

The *center* $Z(G)$ of a group is the set $\{z \in G : (\forall g \in G)\, gz = zg\}$.

Exercise E1.10. Prove the following results:

(i) The center of a Hausdorff topological group is closed.
[Hint. Define $Z(g,G) = \{z \in G : zg = gz\}$. Define $c_g: G \to G$ by $c_g(z) = zgz^{-1}g^{-1}$. Then $Z(g,G) = c_g^{-1}(1)$ and $Z(G) = \bigcap_{g\in G} Z(g,G)$.]

(ii) Let X be an arbitrary set and $T \cong (\mathbb{R}/\mathbb{Z})^X$ a torus which is contained as a normal subgroup in a connected topological group G. Then T is central, that is, all of its elements commute with all elements of G.
[Hint. Consider in T the subgroup S of all elements of finite order. Every automorphism of T maps S into itself, and thus S is normal in G. But S is contained in $(\mathbb{Q}/\mathbb{Z})^X$ and this is a totally disconnected subgroup. Hence S is totally disconnected. By 1.32, S is central, that is $S \subseteq Z(G)$. Also, S is dense in T. Conclude that T is central.] □

Chapter 2
The neighborhood filter of the identity

We have seen above that many properties of topological groups may be expressed in terms of the filter $\mathfrak{U} = \mathfrak{U}(1)$ of neighborhoods of the identity.

We recall that a topology $\mathfrak{O}$ on a set may be described by a function $x \mapsto \mathfrak{U}(x)$ which associates with each point a filter of subsets of X such that the following conditions are satisfied:

(U1) $\big(\forall x \in X,\, U \in \mathfrak{U}(x)\big)\, x \in U$.

(U2) $\big(\forall x \in X, U \in \mathfrak{U}(x)\big)\big(\exists V \in \mathfrak{U}(x)\big)\,(\forall v \in V)\, U \in \mathfrak{U}(v)$.

Condition (U2) can be expressed in equivalent form as follows:

(U) $(\forall x \in X, U \in \mathfrak{U}(x))(\exists V) x \in V \subseteq U$ *and* $(\forall v \in V)\, V \in \mathfrak{U}(v)$.

Notice that in (U2) the last statement is $U \in \mathfrak{U}(v)$ while in (U) it reads $V \in U(v)$.

The relation between the validity of (U1) and (U2), equivalently, of (U), and the existence of $\mathcal{O}$ is as follows:

Theorem 2.1. *Let X be a nonempty set such that for each point $x \in X$ there is a filter $\mathfrak{U}(x)$ of subsets of X, then the following conditions are equivalent:*

(i) *There is a unique topology $\mathcal{O}$ on X such that $\mathfrak{U}(x)$ is the neighborhood filter of x for each $\in X$.*

(ii) *For each $x \in X$ the filter $\mathfrak{U}(x)$ satisfies the conditions* (U1) *and* (U2).

(iii) *For each $x \in X$ the filter $\mathfrak{U}(x)$ satisfies the condition* (U) *If conditions* (ii) *or* (iii) *are satisfied, then a set $U \subseteq X$ is a member of $\mathcal{O}$ iff* $(\forall u \in U)\ U \in \mathfrak{U}(u)$. □

[For a proof see e.g. the Lecture Notes of "Introduction to Topology," Summer 2005, Theorem 1.15.]

Notice that when the conditions of Theorem 2.1 are satisfied, then for each subset $S \subseteq X$, the set $\{s \in S : S \in \mathfrak{U}(s)\}$ is the interior of S, that is, the largest $\mathcal{O}$-open subset contained in S.

If G is a group then the single filter $\mathfrak{U}$ suffices to characterize a group topology of G; and we want to discuss this now.

Proposition 2.2. *Let $\mathfrak{U}$ be a filter on a group such that the following conditions are satisfied:*

(V1) $(\forall U \in \mathfrak{U})\ 1 \in U$,

(V2) $(\forall U \in \mathfrak{U})(\exists V \in \mathfrak{U})\ V^2 \stackrel{\text{def}}{=} VV \subseteq U$. *Then there is a unique topology $\mathcal{O}$ on G such that the neighborhood filter of $g \in G$ is $g\mathfrak{U} = \{gU : U \in \mathfrak{U}\}$.*

Moreover, with respect to this topology, all left translations $x \mapsto gx$ are homeomorphisms of G; in particular, G is homogeneous. Also, multiplication is continuous at $(1, 1)$.

Proof . We verify conditiond (U1) and (U2) for the filters $\mathfrak{U}(g) = g\mathfrak{U}$, and then invoke Theorem 2.1 for the existence and uniquelness of $\mathfrak{O} = \{U \subseteq X : (\forall u \in U) U \in g\mathfrak{U}\}$.

So we let $x \in G$ and $U \in \mathfrak{U}(x) = x\mathfrak{U}$. Then $x^{-1}U \in \mathfrak{U}$. Thus by (V2) there is a $W \in \mathfrak{U}$ such that $W^2 \subseteq x^{-1}U$. Let $V = xW$ and $v \in V$. Then $x = x1 \in xW \subseteq x(x^{-1}U) = U$, and so (U1) holds. Also, $v = xw$ with $w \in W$. Then $vW = xwW \subseteq x(x^{-1}U) = U$ and since $vW \in v\mathfrak{U} = \mathfrak{U}(v)$ we have $U \in \mathfrak{U}(v)$. Therefore (U2) holds, and so 2.1 applies.

The functions $x \mapsto gx$ permute the filters $h\mathfrak{U} = \mathfrak{U}(h)$, $h \in G$ and thus are homeomorphisms w.r.t. $\mathfrak{O}$. It is clear from (V2) that multiplication is continuous at $(1, 1)$. □

Condition 2.2(V2) is equivalent to
(V2′) $(\forall U \in \mathfrak{U})(\exists V, W \in\ U)\ VW \subseteq U$.
If we define $\mathfrak{U}^2 = \{S \subseteq X : (\exists V, W \in \mathfrak{U})\, VW \subseteq S\}$, then condition 2.2(V2) can be expressed in the form
(ii″) $\mathfrak{U} = \mathfrak{U}^2$.

Let us call the unique topology introduced on G in Proposition 2.2 the *left canonical topology.*

Theorem 2.3. *For a group G and a filter $\mathfrak{U}$ on G satisfying the conditions* (V1) *and* (V2) *of* Proposition 2.2. *Then the following conditions are equivalent:*
(i) *The left canonical topology makes G into a topological group.*
(ii) *In addition to* 2.2(V1,V2) *the following conditions hold:*
(V3) $(\forall U \in \mathfrak{U})(\exists V \in \mathfrak{U})\ V^{-1} \subseteq U$.
(V4) $(\forall g \in G)(\forall U \in \mathfrak{U})(\exists V \in \mathfrak{U})\ gVg^{-1} \subseteq U$.
If these conditions are satisfied, then for all $g \in G$ one has $g\mathfrak{U} = \mathfrak{U}g = \mathfrak{U}(g)$.

Proof . Condition (V4) says that $\mathfrak{U} \subseteq g\mathfrak{U}g^{-1}$ for all $g \in G$. This holds for g^{-1} in place of g and thus $\mathfrak{U} \subseteq g^{-1}\mathfrak{U}g$ and so $g\mathfrak{U}g^{-1} \subseteq \mathfrak{U}$. Hence $\mathfrak{U} = g\mathfrak{U}g^{-1}$ and so

$$(*) \qquad g\mathfrak{U} = \mathfrak{U}g.$$

Now let $(g, h) \in G \times G$ and show that multiplication is continuous at (g, h). If $U \in \mathfrak{U}$ find $W \in \mathfrak{U}$ such that $W^2 \subseteq U$ and then find $V \in \mathfrak{U}$ so that $V \subseteq hWh^{-1}$. Then $(gV)(hW) \subseteq ghWh^{-1}hW = ghW^2 \subseteq ghU$, and this shows continuity of multiplication at (g, h). Remains to show the continuity of $x \mapsto x^{-1}$ at g, say. But by $(*)$ we have $(g\mathfrak{U})^{-1} = \mathfrak{U}^{-1}g^{-1} = \mathfrak{U}g^{-1} = g^{-1}\mathfrak{U}$ since $\mathfrak{U}^{-1} = \mathfrak{U}$ by (V3). □

Thus, by Theorem 2.3, in a topological group, the left canonical and the right canonical topologies agree.

For a function f, for a filter basis $\mathfrak{B}$, we define the filter basis $f(\mathfrak{B})$ to be the set $\{f(B) : B \in \mathfrak{B}\}$. Then Conditions (ii) and (iii) in Theorem 2.3 may be reformulated, in an equivalent way, as follows:
(V2′) $\mathfrak{U}^{-1} = \mathfrak{U}$
(V3′) $(\forall g \in G)\ g\mathfrak{U}g^{-1} = \mathfrak{U}$.

We observe that (V1) is a consequence of (V2) and (V3): Let $U \in \mathfrak{U}$; then by (V2) there is a $V \in \mathfrak{U}$ such that $VV \subseteq U$. By (V3), $V^{-1} \in \mathfrak{U}$. Since $\mathfrak{U}$ is a filter, $W \overset{\text{def}}{=} V \cap V^{-1}$ is an element of $\mathfrak{U}$ and thus $W \neq \emptyset$. Let $w \in W$. Then

$$1 = ww^{-1} \in WW^{-1} \subseteq VV \subseteq U.$$

This proves (V1).

Summary. *Given a group G, we have a bijection between the set of all group topologies on G and the set of all filters $\mathfrak{U}$ satisfying the conditions* (V2),(V3), *and* (V4).

Exercise E2.1. Prove the following
Proposition. *Let G and H be topological groups with their filters $\mathfrak{U}_G$ and $\mathfrak{U}_H$ of identity neighborhoods, respectively. Then a morphism $f: G \to H$ is both continuous and open iff $f(\mathfrak{U}_G) = \mathfrak{U}_H$.* □

Groups Generated by Local Groups

This section deals with generating groups from local data in topological groups. Dealing with local topological groups is always messy. It is unfortunate that each author has a definition different from all other ones. The situation is a little better in the case of the idea of a local group *within a given group.* It is this situation we are dealing with here. In fact we shall consider a group G and a subset K supporting a topology τ_K satisfying the following conditions, to be augmented as we proceed:

(i) $\mathbf{1} \in K$.
(ii) $(\forall x, y \in K,\ V \in \tau_K) \quad xy \in V \Rightarrow (\exists U \in \tau_K)\ y \in U$ and $xU \subseteq V$.
(iii) The set $D \stackrel{\text{def}}{=} \{(x, y) \in K \times K \mid xy \in K\}$ is a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $K \times K$, and multiplication $(x, y) \mapsto xy : D \to K$ is continuous at $(\mathbf{1}, \mathbf{1})$.
(iv) $K^{-1} = K$.
(v) Inversion $x \mapsto x^{-1}: K \to K$ is continuous at $\mathbf{1}$.
(vi) $(\forall y \in K,\ V \in \tau_K) \quad y \in V \Rightarrow (\exists U \in \tau_K)\ \mathbf{1} \in U$ and $Uy \subseteq V$.

We define a subset of the set of subsets of G as follows:

$$\tau_G = \{W \subseteq G \mid (\forall w \in W)(\exists U \in \tau_K)\ \mathbf{1} \in U \text{ and } wU \subseteq W\}.$$

It follows immediately from the definition that τ_G is a topology on G and that it is invariant under all left translations, i.e. that all left translations L_g, $L_g(x) = gx$ are τ_G-homeomorphisms. If we apply (ii) with $y = 1$ and consider the definition of τ_G we obtain at once that every $V \in \tau_K$ is a member of τ_G:

$$(\tau) \qquad \tau_K \subseteq \tau_G.$$

In particular, $K \in \tau_G$ (i.e. K is open in G) and $\tau_G|K = \tau_K$ (i.e. the topology of G induces the given one on K).

Lemma 2.4. *Assume that G is a group and that $K \subseteq G$ satisfies conditions* (i), …, (vi). *Then there is a unique maximal τ_G-open subgroup H of G such that $(H, \tau_G|H)$ is a topological group. In particular the connected component G_0 of $\mathbf{1}$ in G is topological, and if K is connected, then G_0 is the subgroup $\langle K \rangle$ generated by K.*

Proof. Since $\tau_K \subseteq \tau_G$ multiplication and inversion of G are continuous at $(\mathbf{1}, \mathbf{1})$ and $\mathbf{1}$, respectively, by (iii) and (v).

As a first step we shall construct H. Let $\mathcal{U}$ denote the neighborhood filter of the identity in (G, τ_G). The group G acts on the set of all filters $\mathcal{F}$ on G via $(g, \mathcal{F}) \mapsto g\mathcal{F}g^{-1} = \{gFg^{-1} \mid F \in \mathcal{F}\}$. We set $H = \{g \in G \mid g\mathcal{U}g^{-1} = \mathcal{U}\}$, the stabilizer of $\mathcal{U}$ for this action. Then H is a subgroup. By (iii) there is an identity neighborhood $U \in \tau_G$ such that $UU \subseteq K$. By (iii) once more we find an identity neighborhood $V \in \tau_G$ such that $VV \subseteq U$ and by (v) we find a $W \in \tau_G \cap \mathcal{U}$ such that $W \cup W^{-1} \subseteq V$. As a consequence we have $WWW^{-1} \subseteq K$. Thus the function $y \mapsto yw^{-1}: W \to K$ is defined and by (vi) it is continuous at $\mathbf{1}$. As a consequence, the function $x \mapsto wxw^{-1}: W \to K$ is defined for all $w \in W$ and is continuous at $\mathbf{1}$, since all left translations are continuous. As a consequence, $w\mathcal{U}w^{-1} = \mathcal{U}$. Thus $W \subseteq H$. Therefore H contains all hW, $h \in H$ and thus is open. By definition, all inner automorphisms I_h of H, $I_h(x) = hxh^{-1}$ are continuous. Then H is a group in which left translations and all inner automorphisms are continuous, multiplication is continuous at $(\mathbf{1}, \mathbf{1})$ and inversion is continuous at $\mathbf{1}$. We claim that a group with these properties is topological: We note that the right translations $R_g = I_{g^{-1}}L_g$ are continuous and that inversion ι, $\iota(x) = x^{-1}$ is continuous at each g because ι is continuous at $\mathbf{1}$ and $\iota = R_{g^{-1}} \circ \iota \circ L_{g^{-1}}$. Finally, multiplication μ, $\mu(x, y) = xy$ is continuous at each (g, h) because μ is continuous at $(\mathbf{1}, \mathbf{1})$ and $R_h \circ L_g \circ \mu \circ (L_{g^{-1}} \times R_{h^{-1}}) = \mu$.

As a second step we show that H is the largest open topological subgroup of G. Let A be a subgroup of G which is τ_G-open and is topological with respect to $\tau_G|A$. Since A is open, the neighborhood filter $\mathcal{U}_A$ of the identity in A generates $\mathcal{U}$. If $a \in A$, since $(A, \tau_G|A)$ is topological, $a\mathcal{U}_A a^{-1} = \mathcal{U}_A$, and thus $a\mathcal{U}a^{-1} = \mathcal{U}$. Then $a \in H$ by the definition of H. Therefore $A \subseteq H$.

Thirdly we observe that $G_0 \subseteq H$. Since H is open, this will be shown if we prove that every open subgroup U of G is also closed and thus must contain the identity component. Now each left translations L_g of (G, τ_G) is continuous and thus, having the inverse $L_{g^{-1}}$, is a homeomorphism. Hence gU is open for all $g \in G$. Thus $U = G \setminus \bigcap_{g \notin U} gU$ is closed.

Finally assume that K is connected. Then K contains $\mathbf{1}$ by (i) and is connected as a subspace of (G, τ_G) since $\tau_G|K = \tau_K$. Hence $K \subseteq G_0$ and thus $\langle K \rangle \subseteq G_0$. From $\tau_K \subseteq \tau_G$ we know that K is open in (G, τ_G). Hence $K^n = \bigcup_{k_1, \ldots, k_{n-1} \in K} k_1 \cdots k_{n-1} K$ is open. Since $K^{-1} = K$ we have $\langle K \rangle = \bigcup_{n \in \mathbb{N}} K^n$ and so this group is open. Then $\langle K \rangle$ contains G_0 as we have seen in the previous paragraph. Hence $G_0 = \langle K \rangle \subseteq H$. □

We summarize the essence of this discussion in the following theorem.

Theorem 2.5. (Groups Generated by Local Subgroups) *Let K be a symmetric subset ($K = K^{-1}$) of a group G containing $\mathbf{1}$. Assume that K is a connected topological space such that*

(i) *x, y, $xy \in K$, with $xy \in V$ for an open subset V of K imply the existence of open neighborhoods U_x and U_y of x and y such that $xU_y \cup U_x y \subseteq V$,*
(ii) *$\{(x, y) \in K \times K \mid x, y, xy \in K\}$ is a neighborhood of $(\mathbf{1}, \mathbf{1})$ in $K \times K$, and multiplication is continuous at $(\mathbf{1}, \mathbf{1})$,*
(iii) *inversion is continuous at $\mathbf{1}$.*

Then there is a unique topology on the subgroup $\langle K \rangle$ generated by K which induces on K the given topology and makes $\langle K \rangle$ a topological group such that K is an open identity neighborhood of $\langle K \rangle$. □

Recall that in these circumstances, $\langle K\rangle$ is contained in the unique largest open subgroup H of (G, τ_G) which is a topological group $(H, \tau_G|H)$.

Corollary 2.6. (Generating Subgroups of Topological Groups) *Let G be a topological group and K a symmetric connected subspace containing the identity such that the following condition is satisfied:*

(∗) *x, y, $xy \in K$, with $xy \in V$ for an open subset V of K imply the existence of open neighborhoods U_x and U_y of x and y such that $xU_y \cup U_x y \subseteq V$. Also, $\{(x,y) \in K \times K : xy \in K\}$ is a neighborhood of $(1,1)$ in $K \times K$.*

Then there is a topological group A and an injective morphism of topological groups $f\colon A \to G$ such that for some open symmetric identity neighborhood V of A we have

(i) *$A = \langle V\rangle$,*

(ii) *$f(V) = K$ and $f|V\colon V \to K$ is a homeomorphism.*

Hypothesis (∗) is satisfied if there is an open symmetric identity neighborhood U in G and K is a connected symmetric subset of U containing the identity such that

$$KK \cap U \subseteq K.$$

Proof. The hypotheses of 2.5 are quite clearly satisfied. Hence the subgroup $\langle K\rangle$ has a unique topology τ making it into a topological group H and inducing on K the same topology as does that of G such that $V \stackrel{\text{def}}{=} (K, \tau|K)$ is open and generates H. The inclusion map $f\colon H \to G$ then satisfies the requirements.

The last claim of the corollary is straightforward. □

Notation. A subset K of a topological group G is called a *local subgroup* of G if it is symmetric and there is a symmetric open identity neighborhood U of G such that

$$KK \cap U \subseteq K.$$

Exercise E2.2. Provide all details of the proof of Corollary 2.6: Check carefully that all hypotheses of Theorem 2.5 are satisfied. In particular, verify that a local subgroup K of G satisfies (∗).

One could reformulate Corollary 2.6 by saying that

a connected local subgroup K of a topological group generates a subgroup $\langle K\rangle$ which has in addition to the induced group topology one that in general is finer such that both induce on K the same topology and such that K is open for the finer one.

Metrizability of Topological Groups

A metric d on a group G is called *left invariant* if $d(gx, gy) = d(x, y)$ for all g, x, $y \in G$.

We may cast the presence of a left invariant metric into different guises involving functions.

Lemma 2.7. *For any Hausdorff topological group G the following statements are equivalent.*

(i) *There exists a left invariant metric d on G defining the topology of G.*

(ii) *There exists a continuous function $\|\cdot\|: G \to \mathbb{R}^+ = [0, \infty[$ such that*

 (1) $\|x\| = 0$ *if and only if* $x = 1$.

 (2) $\|x^{-1}\| = \|x\|$ *for all* $x \in G$.

 (3) $\|xy\| \le \|x\| + \|y\|$ *for all* $x,\ y \in G$.

 (4) *For each identity neighborhood U there is an $n \in \mathbb{N}$ such that* $\|g\| < \frac{1}{n}$ *implies* $g \in U$.

(iii) *There exists a function $p: G \to [0, 1]$ such that*

 (1) $p(1) = 0$ *and for each identity neighborhood U there is an $n \in \mathbb{N}$ such that* $p(g) < \frac{1}{n}$ *implies* $g \in U$.

 (2) *For all $n \in \mathbb{N}$ there is an identity neighborhood U such that for all $g \in G$ and $u \in U$ the relation* $p(gu) \le p(g) + \frac{1}{n}$ *holds.*

If these conditions are satisfied, then $\|\cdot\|$ may be chosen to arise from d, p from $\|\cdot\|$, and d from p, as follows.

$$\begin{aligned} \|x\| &= d(x, 1), \\ p(x) &= \min\{\|x\|, 1\}, \\ d(x, y) &= \sup\{|p(gy) - p(gx)| : g \in G\}. \end{aligned}$$

Proof. (i)⇒(ii). Set $\|x\| \stackrel{\text{def}}{=} d(x, 1)$. Then (ii)(1) follows from the positive definiteness of the metric. Further $\|x^{-1}\| = d(x^{-1}, 1) = d(xx^{-1}, x) = d(1, x) = d(x, 1) = |x|$ by left invariance and symmetry. Thus (ii)(2) holds. Finally, $|xy| = d(xy, 1) = d(y, x^{-1}) \le d(y, 1) + d(1, x^{-1}) = d(y, 1) + d(x^{-1}, 1) = \|y\| + \|x^{-1}\| = \|x\| + \|y\|$ by left invariance, triangle inequality, and (ii)(2). This shows (ii)(3) holds and (ii)(4) is trivial.

(ii)⇒(iii). Set $p(x) = \min\{\|x\|, 1\}$ for all $x \in G$. Then $p(1) = 0$ is clear. By (ii)(4), for every identity neighborhood U there is an $n \in \mathbb{N}$ such that $\|g\| < \frac{1}{n}$ implies $g \in U$. This is (iii)(1). Next (ii)(3) and the continuity of $\|\cdot\|$ give (iii)(2).

(iii)⇒(i). Set $d(x, y) \stackrel{\text{def}}{=} \sup\{|p(gy) - p(gx)| : g \in G\}$. Since p is bounded, there is no problem with the existence of the least upper bound. Then $d(x, y) \ge 0$ and $d(x, y) = 0$ iff $(\forall g \in G)\ p(gy) = p(gx)$, and this holds only if $0 = p(1) = p(y^{-1}y) = p(y^{-1}x)$ for all $xy \in G$. By (iii)(1), this implies $y^{-1}x = 1$; since G is Hausdorff, this implies $x = y$, since then 1 is the only element contained in each identity neighborhood. Conversely, if $x = y$ then trivially $d(x, y) = 0$. Hence d is definite. The symmetry of d is immediate from the definition.

Also $d(gx, gy) = \sup\{|p(hgx) - p(hgy)| : h \in G\} = d(x, y)$. Thus d is left invariant.

Finally $|p(gx) - p(gz)| \le |p(gx) - p(gy)| + |p(gy) - p(gz)| \le d(x, y) + d(y, z)$ for all $g \in G$ whence $d(x, z) \le d(x, y) + d(y, z)$. Thus d is indeed a left invariant metric. It remains to show that d defines the topology. Because of left invariance, it suffices to show that the sequence of sets $U_n \stackrel{\text{def}}{=} \{x \in G \mid d(x, 1) < \frac{1}{n}\}$ for $n = 1, 2, \ldots$

forms a basis for the filter of identity neighborhoods. First we show that all U_n are identity neighborhoods. Assume that n is given. By(iii)(2) there is an identity neighborhood $U = U^{-1}$ such that for all $g \in G$ we have $p(gu) \leq p(g) + \frac{1}{2n}$ and $p(g) = p(guu^{-1}) \leq p(gu) + \frac{1}{2n}$ for all g whence $|p(gu) - p(g)| \leq \frac{1}{2n}$ for all $g \in G$, $u \in U$ and thus $d(u, 1) \leq \frac{1}{2n} < \frac{1}{n}$. Thus $U \subseteq U_n$. Now let an open identity neighborhood U be given. Then by (iii)(1) we find an $n \in \mathbb{N}$ such that $x \notin U$ implies $\frac{1}{n} \leq p(x) = |p(1x) - p(1)| \leq \sup\{|p(gx) - p(g)| : g \in G\} = d(x, 1)$. □

Lemma 2.8. *Assume that d, $\|\cdot\|$, and p are linked as in Lemma* 2.7 *and that Γ is a set of automorphisms of the topological group G. Then the following conditions are equivalent:*

(4) $\|\gamma(x)\| = \|x\|$ *for all* $x \in G$, $\gamma \in \Gamma$,
(4′) $p\big(\gamma(x)\big) = p(x)$ *for all* $x \in G$, $\gamma \in \Gamma$,
(4″) $d\big(\gamma(x), \gamma(y)\big) = d(x, y)$ *for all* $x \in G$, $\gamma \in \Gamma$.

If Γ is the group of inner automorphisms, then these conditions are also equivalent to

(4‴) $d(xg, yg) = d(x, y)$ *for all* g, x, $y \in G$.

Proof. The proofs of (4)⇒(4′)⇒(4″)⇒(4) are straightforward from the definitions.

Assume now that Γ is the group of inner automorphisms. We note that $d(xg, yg) = d(g^{-1}xg, g^{-1}yg)$ by left invariance. Thus invariance of the metric under right translations and invariance under inner automorphisms are equivalent for any left invariant metric. □

Condition (4‴) is equivalent to the additional *right invariance* of d. A metric which is both left and right invariant is called *biinvariant*.

In conjunction with Lemma 2.7, a left invariant metric defining a topology can also be translated into terms of certain families of identity neighborhoods.

Lemma 2.9. *For any topological group G, the following condition is also equivalent to the conditions* (i). (ii) *and* (iii) *of* Lemma 2.7.

(iv) There is a function $r \mapsto U(r) \colon]0, \infty[\to \mathcal{P}(G)$ into the set of subsets of G containing 1 such that the following conditions are satisfied:

(A) $(\forall r > 1)\, U(r) = G$.
(B) $(\forall 0 < s) \bigcup_{r<s} U(r) = U(s)$.
(C) For each identity neighborhood U there is an $n \in \mathbb{N}$ such that $U(\frac{1}{n}) \subseteq U$.
(D) For each $n \in \mathbb{N}$ there is an identity neighborhood U such that $U(r)U \subseteq U(r + \frac{1}{n})$ holds.

Moreover, the two concepts p of 2.7(iii) and $U(\cdot)$ are related by

$$p(g) = \inf\{r \in]0, 1] \mid g \in U(r)\} \quad \text{and} \quad U(r) = \{g \in G \mid p(g) < r\}.$$

Proof . 2.7(iii)⇒(iv) For $0 < r$ define $U(r) \overset{\text{def}}{=} \{g \in G \mid p(g) < r\}$. Let $r > 1$. Then for all $g \in G$ we have $p(g) \leq 1 < r$ and so $g \in U(r)$. Now (A) follows from the fact that $p(g) \leq 1$ for all $g \in G$.

Proof of (B). Let $g \in \bigcup_{r<s} U(r)$. Then there is an $r < s$ such that $g \in U(r)$ and then by definition $p(g) < r$. Then $p(g) < s$, i.e. $g \in U(s)$. Now let, conversely, $g \in U(s)$. Then $p(g) < s$ by definition. Set $t = \frac{p(g)+s}{2}$. Then $p(g) < t < s$, and thus $g \in U(t) \subseteq \bigcup_{r<s} U(r)$.

Proof of (C). For a given U choose n as in 2.7(iii)(1). Then $g \in U(\frac{1}{n})$ implies $p(g) < \frac{1}{n}$ and thus $g \in U$.

Proof of (D). By 2.7(iii)(2) for a given $n \in \mathbb{N}$ we find an identity neighborhood such that $p(gu) < p(g) + \frac{1}{n}$ for all $g \in G$ and $u \in U$. So for a $g \in U(r)$ and $u \in U$ we have $p(gu) < t_{g,u} < p(g) + \frac{1}{n} < r + \frac{1}{n}$ so that $gu \in \bigcup_{s<r+\frac{1}{n}} U(s) = U(r + \frac{1}{n})$.

Finally, p is retrieved from $U(\cdot)$ via $p(g) = \inf\{r \in]0, 1] \mid g \in U(r)\}$; indeed let the right side be denoted by m. If $g \in U(r)$, then by definition $p(g) < r$, and so $p(g)$ is a lower bound for the set $\{r \mid g \in U(r)\}$. Hence $p(g) \leq m$. Now let $p(g) < r$. Then $g \in U(r)$ and thus $m \leq r$. It follows that $m \leq p(g)$ and $p(g) = m$ is proved.

(iv)⇒2.7(iii) For $g \in G$ define $p(g) = \inf\{r \in]0, 1] \mid g \in U(r)\}$. This definition is possible by (A). Clearly, $0 \leq p(g) \leq 1$. Since $1 \in U(r)$ for all $r > 0$ by hypothesis on $U(\cdot)$, we have $p(1) = 0$.

Proof of (C)⇒(iii)(1). Let U be given. Find n so that $U(\frac{1}{n}) \subseteq U$. If $p(g) < \frac{1}{n}$, then $g \in U(\frac{1}{n}) \subseteq U$. Proof of (D)⇒(iii)(2). Let $n \in \mathbb{N}$. Then by (D) there is an identity neighborhood U such that $U(r)U \subseteq U(r + \frac{1}{n})$. Now let $g \in G$ and $u \in U$. Take any r with $g \in U(r)$. Then $gu \in U(r)U \subseteq U(r + \frac{1}{n})$ and thus $p(gu) < r + \frac{1}{n}$. We conclude $p(gu) \leq p(g) + \frac{1}{n}$.

Finally, $U(\cdot)$ is retrieved from p via $U(r) = \{g \in G \mid p(g) < r\}$. Indeed, let $g \in U(r)$, then by (B) there is an $s < r$ with $g \in U(s)$. Then $p(g) \leq s < r$; thus the left hand side is contained in the right hand side. Conversely, assume that $p(g) < r$. Since $p(g) = \inf\{s \mid g \in U(s)\}$, there is an s with $p(g) \leq s < r$ such that $g \in U(s)$. Then, a fortiori, $g \in U(r)$. So both sides are equal. □

Lemma 2.10. *The metric d corresponding to the p in Lemma* 2.9 *is biinvariant if and only if $gU(r)g^{-1} = U(r)$ for all $g \in G$ and all $r \in [0, 1]$. More generally, d is invariant under the members of a set Γ of automorphisms of G if and only if all sets $U(r)$ are invariant under the automorphisms from Γ.*

Proof . This is immediate from 2.8 and the connection between $r \mapsto U(r)$ and p in 2.9. □

The function $U(\cdot)$ now permits an access to metrizability theorems on a purely algebraic level. A subset D of a set X endowed with a partial order $\leq$ is called *a directed set* if it is not empty and each nonempty finite subset of D has an upper bound in D.

Definition 2.11. A *semigroup with a conditionally complete order* is a semigroup S together with a partial order $\leq$ such that the following conditions are satisfied:
(i) $(\forall s,\, t,\, x)\ s \leq t \Rightarrow sx \leq tx$, and $xs \leq xt$.
(ii) $(\forall s,\, t)\ s \leq st$.
(iii) Every directed subset of S has a least upper bound. Further S has a (semigroup) zero which is the largest element of S. □

The set of identity neighborhoods of a topological group G is a semigroup with the conditionally complete order $\subseteq$.

Lemma 2.12. *Let S be a semigroup with a conditionally complete order. Assume that there is a sequence of elements u_n, $n = 1, 2 \ldots$ in S satisfying the following condition: $(\surd)$ $u_{n+1}^2 \leq u_n$.*
Then there is a function $F\colon]0, \infty] \to S$ such that
(I) $(\forall r > 1)\ F(r) = \max S$.
(II) $(\forall 0 < s)\ \sup_{r<s} F(r) = F(s)$.
(III) $(\forall n \in \mathbb{N})\ F(\frac{1}{2^n}) \leq u_n$, *and*
(IV) $(\forall r > 0,\ n \in \mathbb{N})\ F(r)u_{n+1} \leq F(r + \frac{1}{2^n})$.
Moreover, F takes its values in the smallest subsemigroup containing

$$\{u_1, u_2, \ldots; \max S\}$$

which is closed under the formation of directed suprema.

Proof . (a) Note that $u_{n+1} \leq u_{n+1}^2 \leq u_n$ by 2.11(ii) and $(\surd)$. Thus

$$(\#) \qquad (\forall m,\, n \in \mathbb{N},\, m \leq n) \quad u_n \leq u_m.$$

We shall first define a function $f\colon J \to S$ on the set J of dyadic rationals $r = m/2^n$, $m,\, n \in \mathbb{N}$ with values in the subsemigroup $T \stackrel{\text{def}}{=} \langle u_1, u_2, \ldots; \max S\rangle$. Once and for all we set $f(r) = \max S \in T$ for all $1 \leq r \in J$. To get started in earnest, we set $f(1/2) = u_1 \in T$. The next step is to define $f(r)$ for $r \in \{\frac{1}{4}, \frac{3}{4}\}$; note that $f(\frac{2}{4}) = u_1$ is already defined. We set $f(\frac{1}{4}) = u_2 \in T$ and $f(\frac{3}{4}) = f(1/2)u_2 = u_1u_2 \in T$. This indicates our strategy of producing a recursive definition. We set

$$J_n = \left\{\frac{m}{2^n} \mid m = 1, \ldots, 2^n\right\}, \quad n = 0, 1, 2, \ldots$$

and note $J_0 = \{1\} \subseteq J_1 = \{\frac{1}{2}, 1\} \subseteq J_2 = \{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}\} \subseteq J_3 \subseteq \cdots$ and $J = (J \cap [1, \infty[) \cup \bigcup_{n\in\mathbb{N}} J_n$. Assume that f is defined on J_n with $f(J_n) \subseteq T$ in such a way that $f(\frac{1}{2^m}) = u_m$ for $m = 1, \ldots, n$ and that

$$(\dagger_n) \qquad (\forall r \in J_n)\ f(r)u_n \leq f(r + \frac{1}{2^n})$$

holds. We note that $f(r) \leq f(r)u_m$ by 2.11(ii) and that therefore $(\dagger_n)$ implies that f *is monotone on* J_n, that is

$$(\#\#) \qquad (\forall r, s \in J, r \leq s)\, f(r) \leq f(s).$$

We must define $f(r)$, $r = m/2^{n+1}$. If m is even, then $r \in J_n$ and $f(r) \in T$ is defined. If $r = \frac{1}{2^{n+1}}$, we set $f(r) = u_{n+1} \in T$; if $r \in J_n$ we set $f(r + \frac{1}{2^{n+1}}) = f(r)u_{n+1} \in TT \subseteq T$. We must show that $(\dagger_{n+1})$ holds.
Case 1. $r \in J_n$. Then $f(r)u_{n+1} = f(r + \frac{1}{2^{n+1}})$ by definition.
Case 2. $r \in J_{n+1} \setminus J_n$. Then $r = r_0 + \frac{1}{2^{n+1}}$ with $r_0 \in \{0\} \cup J_n$. Now

$$f(r)u_{n+1} = \left\{ \begin{array}{ll} u_{n+1}u_{n+1} \leq u_n & \text{if } r_0 = 0 \\ f(r_0)u_{n+1}u_{n+1} \leq f(r_0)u_n & \text{if } r_0 > 0 \end{array} \right\} \leq f(r_0 + 1/2^n)$$
$$= f(r_0 + \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}) = f\big(r + (1/2^{n+1})\big)$$

by definition of f on J_{n+1} and 2.11(i), by ($\surd$), and by induction hypothesis $(\dagger_n)$.

The induction is complete, and we have defined $f\colon J \to S$ by recursion in such a fashion that (##)) and $(\dagger_n)$ are satisfied. Now we extend f to a function $F\colon]0, \infty[\to S$ by $F(r) = \max S$ for $r > 1$ and

$$F(r) = \sup_{s \in J,\, s < r} f(s) \text{ for } 0 < r \leq 1.$$

This least upper bound exists by 2.11(iii). If $\overline{T}$ denotes the smallest subsemigroup of S containing T and being closed under the formation of directed sups, then $\operatorname{im} F \subseteq \overline{T}$.

Clearly (I) is satisfied. Proof of (II). We compute

$$\sup_{0<s<r} F(s) = \sup_{0<s<r} \Big(\sup_{u \in J,\, u<s} f(u) \Big) = \sup_{\{(u,s) | u \in J,\, u<s<r\}} f(u) = \sup_{u \in J,\, u<r} f(r) = F(r)$$

since J is order dense in $]0,1]$.

Proof of (III): We have $F(1/2^n) = \sup_{r \in J,\, r < \frac{1}{2^n}} f(r) \leq u_n$ since $r < \frac{1}{2^n}$ implies $f(r) \leq f(\frac{1}{2^n})$ by (##), and $f(\frac{1}{2^n}) = u_n$ by the construction of f.

Proof of (IV). Fix an $n \in \mathbb{N}$ and consider an $r \in]0,1]$. If $1 - (1/2^{n+1}) \leq r$, then $F(r + (1/2^n)) = \max S \geq F(r)u_{n+1}$. So assume that $r < 1 - (1/2^{n+1})$ and let s be the first element of J_{n+1} such that $r \leq s$. Then (##) and the definition of F implies

(α) $$F(r) = \sup_{q \in J,\, q<r} f(q) \leq f(s),$$

and since the element $s + (1/2^{n+1}) = \big(s - (1/2^{n+1})\big) + (1/2^n) < r + (1/2^n)$ belongs to J, we have

(β) $$f\big(s + (1/2^{n+1})\big) \leq F\big(r + (1/2^n)\big).$$

From $(\dagger_{n+1})$ we get

(γ) $$f(s)u_{n+1} \leq f\big(s + (1/2^{n+1})\big).$$

Now (α), (β), and (γ) and 2.11(i) together imply

$$F(r)u_{n+1} \leq f(s)u_{n+1} \leq F\big(r + (1/2^n)\big),$$

and this is what we had to show. The proof of the lemma is now complete. □

Theorem 2.13 (Characterisation of Left Invariant Metrizability). (a) *For a topological group G, the following conditions are equivalent:*

(1) *The topology of G is defined by a left invariant metric.*

(2) *The filter of identity neighborhoods* (*equivalently, that of any point in* G) *has a countable basis.*

(b) *Also, the following conditions are equivalent:*

(3) *The topology of* G *is defined by a biinvariant metric.*

(4) *The filter of identity neighborhoods has a countable basis each member of which is invariant under inner automorphisms.*

(c) *For a locally compact group* G *conditions* (1) *and* (2) *are equivalent to the following condition.*

(5) *There is a countable family of identity neighborhoods intersecting in* $\{1\}$.

Proof . Clearly, (1)⇒(2) and (3)⇒(4).

We assume (2) and show (1). In order to prove (4)⇒(3) at the same time we consider a set Γ of automorphisms of the topological group G, e.g. $\Gamma = \{\mathrm{id}\}$, or the group of all inner automorphisms. Let O_n, $n \in \mathbb{N}$ be a family of Γ-invariant identity neighborhoods which form a basis for the filter of identity neighborhoods. We define recursively a new basis U_n by setting $U_1 = O_1$. Assume that $U_1, \ldots, U_n$ is defined so that all U_m are Γ-invariant and satisfy $U_m^2 \subseteq U_{m-1} \cap O_{m-1}$, $m = 2, 3, \ldots, n$. There is an identity neighborhood V such that $VV \subseteq U_n \cap O_n$. Since the O_m form a basis for the identity neighborhoods there is an index $j(n) \geq n$ such that $O_{j(n)} \in V$. Set $U_{n+1} = O_{j(n)}$. The recursion is complete and yields a basis of Γ-invariant identity neighborhoods U_n with $(U_{n+1})^2 \subseteq U_n$.

Now we let S denote the semigroup of all Γ-invariant identity neighborhoods under multiplication of subsets of G. Containment $\subseteq$ endows S with a conditionally complete order (see 2.10). Then Lemma 2.12 applied with $u_n = U_n$ yields a function $r \mapsto U(r):]0, \infty[\to \mathcal{P}(G)$ such that Conditions 2.12(I)–(IV) are satisfied with U in place of F. We claim that (A)–(D) from 2.9 are satisfied. We have (A)⇔(I) and (B)⇔(II). In order to prove (C) let U be any identity neighborhood. Since the U_k form a basis for the identity neighborhoods we find an m such that $U_m \subseteq U$. We set $n = 2^m$ and see that $U(1/n) = U(1/2^m) \subseteq U_m \subseteq U$ by (III). In order to verify (D) we let n be given. Pick an $m \in \mathbb{N}$ such that $n \leq 2^m$. Then set $U = U_{m+1}$. Then for each $t > 0$ we have $U(t)U = U(t)U_{m+1} \subseteq U\big(t + (1/2^m)\big) \leq U\big(t + (1/n)\big)$ by (IV) and the monotonicity of $s \mapsto U(s)$, secured by (II). Thus $s \mapsto U(s)$ satisfies conditions (A)–(D) of 2.9. Then Lemmas 2.7, 8, 10 show that G has a Γ-invariant metric defining its topology.

It is obvious that (1) implies (5). We now assume that G is locally compact and prove that (5) implies (1). Assume $\{1\} = \bigcap_{n\in\mathbb{N}} U_n$ for a family of identity neighborhoods U_n. We may assume that $\overline{U_n}$ is compact for all n and that $U_{n+1} \subseteq U_n$. Let U be an open identity neighborhood in G. Claim: There is an N such that $U_N \subseteq U$. Suppose not, then $\{\overline{U_n} \setminus U : n \in \mathbb{N}\}$ is a filter basis of compact sets. Its nonempty intersection is contained in $\{\mathbf{1}\}$ on the one hand and in $G \setminus U$ on the other. This contradiction proves the claim. □

We remark that the preceding theorem allows us to conclude that a topological group with a metrizable identity neighborhood is left invariantly metrizable. This is the case if some identity neighborhood is homeomorphic to an open ball in some Banach space.

We notice that in Theorem 2.13 we have proved a little more:

Corollary 2.14. *Assume that G is a topological group and Γ a set of automorphisms. If G has a countable basis of Γ-invariant identity neighborhoods, then the topology of G is defined by a left-invariant metric satisfying $d\big(\gamma(x),\gamma(y)\big) = d(x,y)$ for all automorphisms from the group $\langle\Gamma\rangle$ generated by Γ.* □

Corollary 2.15. *The topology of every compact group with a countable basis of identity neighborhoods is defined by a biinvariant metric.*

Proof . By Corollary 1.28, every compact group has a basis of identity neighborhoods which are invariant under inner automorphisms. The assertion then follows from 2.14 with the group of all inner automorphisms Γ. □

Exercise E2.3. (i) The semigroup of all subsets of a topological group G containing the identity of G contains various subsemigroups which are conditionally complete in the containment order. Examples:

(a) The semigroup of all normal subgroups.

(b) The semigroup of all open closed normal subgroups.

Note that all elements in these semigroups are idempotent. What are the consequences for the metric constructed according to Theorem 2.13 from a countable basis U_n for the filter of identity neighborhoods consisting of open normal subgroups?

(ii) Show that, for a metric group G there is a family $r \mapsto U(r){:}\,]0,\infty[\, \to \mathcal{P}(G)$ which, in addition to the conditions (A)–(D) of Lemma 2.9 also satisfies the following condition:

(E) $(\forall 0 < s,\, t)\ U(s)U(t) \leq U(s+t)$.

[Hint for (ii). Consider $p(x) = \min\{\|x\|, 1\}$ for a function $\|\cdot\|$ satisfying the conditions 2.7(ii)(1)–(4) and define $U(r) = \{g \in G \mid p(g) < r\}$.] □

Regarding Exercise E2.3(ii) it is not known whether a semigroup theoretical proof exists to construct a function F such as in Lemma 2.12 with the additional property that $F(s)F(t) \leq F(s+t)$. In the presence of certain additional conditions such a proof was given in K. H. Hofmann, *A general invariant metrization theorem for compact spaces*, Fundamenta Math. **68** (1970), 281–296.

Exercise E2.4. Recall that a *pseudometric* satisfies all axioms of a metric with the possible exception of the postulate that $d(x,y) = 0$ implies $x = y$. Use the tools at our disposal in order to prove the following result.

In a topological group G let $\{O_n : n \in \mathbb{N}\}$ be any filter basis of identity neighborhoods. Then there is a sequence of identity neighborhoods U_n satisfying $(U_{n+1})^2 \subseteq U_n$ and a continuous left invariant pseudometric d such that for any n the identity neighborhood U_n contains some open d-ball around the identity. □

Notice that in a group with a left invariant pseudometric the set of elements with distance 0 from the identity is a subgroup. Exercise E2.4 shows that the topology of a topological group (Hausdorff or not) can be defined by a family of pseudometrics.

For the structure of locally compact groups it is relevant that compactly generated locally compact groups can be approximated by locally compact separablew metric groups in the following sense. (The Kakutani–Kodaira–Montgomery–Zippin–Theorem.)

Theorem 2.16. *Assume that G is a topological group with a compact symmetric identity neighborhood K which satisfies $G = \langle K \rangle = \bigcup_{n\in\mathbb{N}} K^n$. If a sequence $(W_n)_{n\in\mathbb{N}}$ of identity neighborhoods is given, then there is a closed compact normal subgroup $N \subseteq \bigcap_{n\in\mathbb{N}} W_n$ such that G/N is separable metric locally compact.*

Proof . Exercise. □

Exercise E2.5. Prove Theorem 2.16.
[Hint. Construct a sequence of compact symmetric identity neighborhoods $(V_n)_{n\in N}$ such that
(i) $V_{n+1}^2 \subseteq V_n \cap W_n$.
(ii) $(\forall g \in K)\ gV_ng^{-1} \subseteq V_{n-1}$.

Set $N = \bigcap_{n\in\mathbb{N}} V_n$ and show that N is a compact normal subgroup.
Prove the following general
Lemma a. *If G is a locally compact group and N a closed normal subgroup, then G/N is locally compact.*

Show that if G and N are as in the theorem then G/N has a countable basis for the filter of its identity neighborhoods.

For a proof prove the following topological
Lemma b. *Let $\mathcal{F}$ be a filter basis of closed sets containing a compact set and let $\bigcap \mathcal{F} \subseteq U$ for an open set U, then there is an $F \in \mathcal{F}$ such that $F \subseteq U$.*

Show further that G/N is locally compact and a countable union of compact subsets K^nN/N. By the Metrizabilty Theorem 2.13, first countable groups are metric, so G/N is metric. Note that all K^nN/N are compact metric hence have a countable basis for their topology. So they are separable spaces. So their union is separable.]

The methods we have discussed actually yield more: The general process leading to Theorem 2.13 in fact allows us to prove the following theorem.

Theorem 2.17. *Let G be a topological group and U an open neighborhood of $\mathbf{1}$. Then there is a continuous function $f\colon G \to [0,1]$ which takes the value 0 in $\mathbf{1}$ and the value 1 in all points outside U.*

Proof . Exercise □

Exercise E2.6. Prove Theorem 2.17.

Definition 2.18. A topological space is called *completely regular* if for each point x and each neighborhood U of x there is a continuous function on the space with values in $[0, 1]$ taking the value 0 in x and the value 1 outside U. A space is a $T_{3\frac{1}{2}}$-space if it is a completely regular Hausdorff space.

In view of the homogeneity of a topological group, Theorem 2.17 shows that every topological group is completely regular. This holds in fact without any separation property. For a topological group the separation axioms T_n, $n = 0, 1, 2, 3, 3\frac{1}{2}$ all coincide.

Chapter 3
Open Mapping Theorems

Let us recall from Proposition 1.13, that for a morphism $f: G \to H$ of topological groups the following conditions are equivalent:

(i) *f is open.*

(ii) *For each $U \in \mathfrak{U}(1)$ the image $f(U)$ has a nonempty interior.*

(iii) *There is a basis $\mathfrak{B}$ of identity neighborhoods U such that $f(U)$ has a nonempty interior.*

(iv) *There is a basis of identity neighborhoods U of G such that $f(U)$ is an identity neighborhood of H.*

(v) *For all $U \in \mathfrak{U}_G(1)$ we have $f(U) \in \mathfrak{U}_H(1)$.*

The relevance of the information concerning the openness of morphisms was illustrated in the Canonical Decomposition Theorem 1.14 of morphisms:
A morphism of topological groups $f: G \to H$ with kernel $N = \ker f$ decomposes canonically in the form

$$\begin{array}{ccc} G & \xrightarrow{\quad f \quad} & H \\ {\scriptstyle q}\downarrow & & \uparrow{\scriptstyle \mathrm{j}} \\ G/N & \xrightarrow[\quad f' \quad]{} & f(G), \end{array}$$

where $q: G \to N$ is the quotient morphism given by $q(g) = gN$, $j: f(G) \to H$ is the inclusion morphism, and $f': G/N \to f(G)$ is the bijective morphism of toplogical groups given by $f'(gN) = f(g)$.

The morphism is open if and only if $f(G)$ is open in H and f' is an isomorphism of topological groups, i.e. is continuous and open.

In topological group theory and functional analysis there is a class of theorems in which sufficient conditions are formulated for a surjective morphism of topological groups to be open *automatically*, that is, to be equivalent to a quotiemt morphisms. Such theorems are called “Open Mapping Theorems”. We shall discuss one of them now because it deals with metric groups.

We need to discuss the concept of *completeness* of a topological group; we shall use this concept here only in the case of metrizable groups.

Definition 3.1. (i) A filter $\mathcal{F}$ in a topological space X *converges to* a point $x \in X$ if $\mathfrak{U}(x) \subseteq \mathcal{F}$ for the neighborhood filter $\mathfrak{U}(x)$ of x. Equivalently, this says that for each neighborhood U of x there is a member $F \in \mathcal{F}$ such that $F \subseteq U$.

(ii) We say that a filter $\mathcal{F}$ in a topological group is a *Cauchy filter* if for each $U \in \mathfrak{U} = \mathfrak{U}(1)$ there is an $F \in \mathcal{F}$ such that $FF^{-1} \subseteq U$.

(iii) We say that a topological group is *complete* if every Cauchy filter converges. □

Exercise E3.1. Formulate corresponding concepts for filter bases: What is a convergent filter basis in a topological space; what is a Cauchy filter basis in a topological group? □

This concept of completeness is more familiar for metric spaces; indeed a metric space X is said to be complete if every Cauchy sequence converges.

A filter $\mathcal{F}$ in a metric space is said to be a *Cauchy filter* if for every ε there is an $F \in \mathcal{F}$ such that the diameter $\sup\{d(x,y) : x, y \in F\}$ of F is less than ε.

From the Characterisation Theorem for Left Invariant Metrizability 2.13 we know that a topological group has a (left invariant) metric defining the topology iff it satisfies the First Axiom of Countability, that is, if the filter of identity neighborhoods has a countable basis.

Exercise E3.2. Let X be a metric space. For a sequence $(x_n)_{n\in\mathbb{N}}$ let $\mathcal{F}$ be the filter generated by the filter basis $\big\{\{x_m | m \geq n\} : n \in \mathbb{N}\big\}$. Show that the sequence converges to x in the sense of metric spaces iff the filter $\mathcal{F}$ converges to x. Show that the sequence is a Cauchy sequence if and only if $\mathcal{F}$ is a Cauchy filter.

Let G be a first countable topological group. Choose a left invariant metric defining the topology. Show that a filter $\mathcal{C}$ is a Cauchy filter in the sense of topological groups iff it is a Cauchy filter in the sense of metric space.

Show that a first countable group is complete in the sense of metric spaces iff it is complete in the sense of topological groups □

We summarize: a first countable topological group is *complete* if every Cauchy sequence converges.

We shall now prove a new characterisation of of openness of a morphism in the case of complete metrizable groups.

Theorem 3.2. *For a morphism $f: G \to H$ of (not necessarily Hausdorff) topological groups consider the following conditions:*

(1) f is open.
(2) $(\forall U \in \mathfrak{U}_G)$ $\operatorname{Interior} \overline{f(U)} \neq \emptyset$.
(3) $(\forall U \in \mathfrak{U}_G)$ $\overline{f(U)} \in \mathfrak{U}_H$
(4) $\big(\forall U \in \mathcal{O}(G)\big)\big(\exists U' \in \mathcal{O}(H)\big)$ $f(U) \subseteq U' \subseteq \overline{f(U)}$.
(5) $(\forall U \in \mathcal{O}(G))$ $f(U) \subseteq \operatorname{Interior} \overline{f(U)}$.

Then (2), (3), (4) and (5) are equivalent and are implied by (1). If G is a first countable complete topological group and H is Hausdorff, then they are all equivalent.

Proof . From 1.13 we know that the openness of f is equivalent to $(\forall U \in \mathfrak{U}_G)$ $f(U) \in \mathfrak{U}_H$ and thus (1) implies (2).

(2) ⇒ (3): Let $U \in \mathfrak{U}_G$. Find $V \in \mathfrak{U}_G$ such that $VV^{-1} \subseteq U$. By (2), there is a $v \in \operatorname{Interior} \overline{f(V)}$; hence $1 = vv^{-1} \in \operatorname{Interior} \overline{f(V)} (\operatorname{Interior} \overline{f(V)})^{-1} \subseteq$

$\overline{f(V)}\,\overline{f(V)}^{-1} \subseteq \overline{f(V)f(V)^{-1}}$ (in view of the the continuity of multiplication and inversion) $= \overline{f(VV^{-1})} \subseteq \overline{f(U)}$. This entails $\overline{f(U)} \in \mathfrak{U}_H$.

(3) $\Rightarrow$ (4): Let $U \in \mathcal{O}(G)$. For each $u \in U$ $u^{-1}U \in \mathfrak{U}_G \cap \mathcal{O}$. By (3) we know that $W(u) = \text{Interior}\,\overline{f(u^{-1}U)}$ is an open neighborhood of 1 in H and thus $f(u)W(u) = f(u)\,\text{Interior}\,\overline{f(u^{-1}U)} = \text{Interior}\,\overline{f(u)f(u^{-1}U)} = \text{Interior}\,\overline{f(U)}$ is an open neighborhood of $f(u)$ in H. If we set $U' = \text{Interior}\,\overline{f(U)}$, then $f(U) \subseteq U' \subseteq \overline{f(U)}$.

(4) $\Rightarrow$ (5) $\Rightarrow$(2) ist trivial.

This was not much harder than the proof of 1.13. We need the additional hypotheses for a proof of (2)$\Rightarrow$(1).

We shall invoke a sequence of lemmas.

The first is a simple consequence of the First Closure Lemma 1.15.

Lemma A. *If X is a subset of a topological group G and if $U \in \mathfrak{U}$, then $\overline{X} \subseteq XU \cap UX$.* □

Lemma B. $(\forall U \in \mathfrak{U})(\exists V = V^{-1} \in \mathfrak{U})\ \overline{VV} \subseteq U$.

Proof . There is a $W \in \mathfrak{U}$ such that $WWW \subseteq U$. By Lemma A we have $\overline{WW} \subseteq WWW$. Thus $\overline{WW} \subseteq U$. Now we set $V = W \cap W^{-1}$; then $V \in \mathfrak{U}$ and $V^{-1} = W^{-1} \cap (W^{-1})^{-1} = V$, and $V \subseteq W$ implies $\overline{VV} \subseteq U$. □

Lemma C. *Let G be a topological group satisfying the First Axiom of Countability. Then there exists a basis $\mathfrak{B} = \{U_n : n = 1, 2, \ldots\}$ of $\mathfrak{U}$ with the following properties:*

(i) $U_n = U_n^{-1}$ (i.e. U_n is symmetric),
(ii) $\overline{U_nU_n} \subseteq U_{n-1}$ für $n = 2, 3, \ldots$

Proof . We construct U_n recursively: Let $\{V_n : n \in \mathbb{N}\}$ be a basis of $\mathfrak{U}$. We set $U_1 = V_1$. Assume that $U_1, \ldots, U_m$ have been constructed such that $U_k \subseteq V_k$. By Lemma B there existists a symmetric neighborhood $U_{n+1} \in \mathfrak{U}$ such that $\overline{U_{n+1}U_{n+1}} \subseteq U_n \cap V_{n+1}$. Then we have $U_{n+1} \subseteq V_{n+1}$ and $\overline{U_{n+1}U_{n+1}} \subseteq U_n$. Thus the construction of the U_n is secured, the properties (i) and (ii) are satisfied and $U_n \subseteq V_n$ implies that the set $\mathfrak{B} = \{U_n : n = 1, 2, \ldots\}$ is a basis of $\mathfrak{U}$. □

Lemma D. *If $\{U_n : n \in \mathbb{N}\}$ is the basis of Lemma C, then*

$$U_mU_{m+1}\ldots U_{m+k} \subseteq U_mU_m, \quad m = 1, 2, \ldots.$$

Proof . Proof by induction with respect to k: By $U_{m+1} \subseteq U_m$ the assertiion is true for $k = 1$. If it is proved for k, then $U_{m+1}\ldots U_{m+k+1} \subseteq U_{m+1}U_{m+1}$ by the induction hypothesis, and $U_{m+1}U_{m+1} \subseteq U_m$ by C(ii). Therefore

$$U_mU_{m+1}\ldots U_{m+k+1} \subseteq U_mU_{m+1}U_{m+1} \subseteq U_mU_m,$$

and this had to be shown. □

Lemma E. *If* $x_k = g_0 \dots g_k$ *such that* $g_j \in U_{m+j}$, $j = 0, \dots, k$, *then* $(x_k)_{k=1,\dots}$ *is a Cauchy sequence in* $\overline{U_m U_m}$. *If* G *is complete, then* $x = \lim x_k \in U_{m-1}$, $m \geq 2$.

Proof. We notice

$$\begin{aligned} x_k^{-1} x_{k+p} &= g_k^{-1} \cdots g_0^{-1} g_0 \cdots g_k g_{k+1} \cdots g_{k+p} \\ &= g_{k+1} \cdots g_{k+p} \in U_{m+k+1} U_{m+k+1} \subseteq U_{m+k} \end{aligned}$$

according to Lemmas D and C(ii). This shows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. If the limit exists, as is the case if G is complete, then $x = \lim x_k \in \overline{U_m U_m} \subseteq U_{m-1}$ by Lemmas D and C(ii). □

Lemma F. *Assume that condition* (4) *of* Theorem 3.2 *is satisfied and let* U'_n *be chosen for* U_n *as in* (4). *Then* $U'_n \subseteq f(U_{n-1})$*—provided that* G *is complete and* H *is Hausdorff.*

Proof. Let $h \in U'_n \subseteq H$. We shall construct recursively elements

(α_p) $$g_p \in U_{n+p}, \quad p = 0, \dots$$

such that the definition

$(*)$ $$x_p = g_0 \cdots g_p \in U_n U_n$$

(see Lemma D) entails the relation

(β_p) $$h^{-1} f(x_p) \in U'_{n+p+1}.$$

The construction is as follows: Since $h \in U'_n \subseteq \overline{f(U_n)}$ by (4), every neighborhood of h in H meets $f(U_n)$. Thus there is an element $g_0 \in U_n$ such that $f(g_0) \in hU'_{n+1}$ since hU'_{n+1} is a neighborhood of h; indeed on account of $\mathbf{1} \in f(U_{n+1}) \subseteq U'_{n+1}$ (by (4)), the element $h = h{\cdot}\mathbf{1}$ is in the open set hU'_{n+1}. Now let $g_0, \dots, g_p$ be constructed as announced. Then $h^{-1} f(x_p) \in U'_{n+p+1}$ where $x_p = g_0 \cdots g_p$, and $U'_{n+p+1} \subseteq \overline{f(U_{n+p+1})}$ by (4). Then every neighborhood of $h^{-1} f(x_p)$ in H meets $f(U_{n+p+1}) = f(U_{n+p+1}^{-1})$ (see Condition C(i)); therefore there exists a

(α_{p+1}) $$g_{p+1} \in U_{n+p+1}$$

such that $f(g_{p+1})^{-1} = f(g_{p+1}^{-1}) \in (U'_{n+p+2})^{-1} h^{-1} f(x_p)$, and thus

(β_{p+1}) $$h^{-1} f(x_{p+1}) = h^{-1} f(x_p) f(g_{p+1}) \in U'_{n+p+2}.$$

Thus the construction is secured in the fashion asserted. Since $g_j \in U_{n+j}$ we know that $(x_p)_{p\in\mathbb{N}_0}$ is a Cauchy sequence in $U_n U_n$ by Lemma E and $(*)$; by the completeness of G it has a limit $x = \lim x_p \in \overline{U_n U_n} \subseteq U_{n-1}$. On the other hand, since $f(x_q) \in hU'_{n+q+1} \subseteq hU'_{n+p+1} \subseteq \overline{hf(U_{n+p+1})}$ for all $p \geq q$, we know $f(x) \in \overline{hf(U_{n+p+1})}$ for all p. If now W is an arbitrary neighborhood of $\mathbf{1}$ in H, then choose $W_1 \in \mathfrak{U}_H$ so that $W_1 W_1 \subseteq W$. By the continuity of f there is a p so that $f(U_{n+p+1}) \subseteq W_1$, since the U_q form a basis of $\mathfrak{U}_G$. Now by Lemma A we have $\overline{f(U_{n+p+1})} \subseteq W_1 W_1 \subseteq W$. Then $f(x) \in hW$ for all $W \in \mathfrak{U}_H$. From the assumption that H is Hausdorff it follows that $f(x) = h$. This shows $h = f(x) \in f(U_{n-1})$. □

Now we conclude that proof of Theorem 3.2.

Proof of (4) ⇒ (1): Since G is complete and H is Hausdorff, Lemma F applies and shows that $f(U_n)$ has a nonempty interior for all n. Since the U_n form a basis of $\mathfrak{U}_G$, Proposition 1.13 shows that f is open. □

In order to produce an Open Mapping Theorem from this result we resort to a definition.

Definition 3.3.a. (i) A topological space X will be called *inexhaustible* if for every countable family of closed sets $A_n \subseteq X$ the relation $X = \bigcup_{n\in\mathbb{N}} A_n$ implies that there is one $n \in \mathbb{N}$ such that $\operatorname{int} A_n \neq \emptyset$. In other words, X is not a countable union of nowhere dense closed subsets.

(ii) A space is called a *Baire space* if every countable union of closed sets with empy interior has an empty interior. □

Obviously, a Baire space is inexhaustible, and the class of Baire spaces indeed provides the tangible examples of inexhaustible spaces. This is due to the following theorem, which is proved in introductory topology courses. [N.Bourbaki, Topologie générale, Chap. 9, \$5, n^0 3, Théorème 1.]

The Baire Category Theorem 3.3.b. *Each of the following conditions suffices for a space X to be a Baire space*

(i) *X is a locally compact Hausdorff space*

(ii) *Each point of x has a closed neighborhood which is a complete metric space with respect to a a metric defining the topology.*

Proof . Let W_n, $n = 1, 2, \ldots$ be a sequence of dense open subsets of X. We claim that $D = \bigcap_{n=1}^{\infty} W_n$ is dense. For a proof let $V \neq \emptyset$ open; we must show $V \cap D \neq emptyset$. We define recursively nonempty open subsets $V_1 = V, V_2, \ldots$. In case (ii), there is no harm in assuming that $\overline{V}$ is completely metrisable. If V_n is defined, since W_n is dense, there is a point in $W_n \cap V_n \neq \emptyset$. This point has a compact, respectively completely metrisable neighborhood V_{n+1} such that $\overline{V_{n+1}} \subseteq V_n \cap W_n$. In the first case we chose V_{n+1} so that $\overline{V_{n+1}}$ is compact, in the second case so that the radius of V_{n+1} is smaller than $\frac{1}{n+1}$.

In Case (i), there is a point $x \in \bigcap_{n=1}^{\infty} \overline{V_{n+1}} \subseteq \bigcap_{n=1}^{\infty} V_n \subseteq \bigcap_{n=1}^{\infty} (V \cap W_n) = V \cap D$.

In Case (ii) we note that $\{V_n : n \in \mathbb{N}\}$ is a Cauchy filter in $\overline{V}$ which has a limit x in $\overline{V}$, contained in every $\overline{V}_n$. Hence $x \in V \cap D$. □

Condition (ii) is satisfied for every complete metric space. Recall that a space is separable if it has a countable dense subset. If a space satisfies the second axiom of countability, that is, its topology has a countable basis, then it is separable and satisfies the first axiom of countability, but the converse fails in general.

Exercise E3.3. Prove the following proposition
In a topological group G the following two conditions are equivalent:
(i) *G satisfies the Second Axiom of Countability.*
(ii) *G satisfies the First Axiom of Countability and is separable.*

[Hint. For a proof of (ii)⇒(i) let D be a countably dense subset and $\{U_n : n \in \mathbb{N}\}$ a countable basis of $\mathfrak{U}$. Show that $\{dU_n : d \in D, n \in \mathbb{N}\}$ is a countable basis of the topology $\mathcal{O}(G)$.]

Theorem 3.4. (Open Mapping Theorem A) *Let $f: G \to H$ be a surjective morphism of topological groups and assume that*
(i) *G is first countable, separable, and complete,*
(ii) *H is inexhaustible and Hausdorff.*
Then f is open.

Proof. By Theorem 3.2 we have to show that for each $U \in \mathfrak{U}_G$ the set Interior $\overline{f(U)}$ is not empty. Let $U \in \mathfrak{U}_G$ and let D be a countable dense subset of G. Then $G = \overline{D} \subseteq DU$ by the First Closure Lemma 1.15(i). Thus, as f is surjective, $H = f(G) = \bigcup_{d\in D} f(d)f(U) \subseteq \bigcup_{d\in D} f(d)\overline{f(U)}$. Since D is countable and H is inexhaustible, there is a $d \in D$ such that Interior $\overline{f(d)f(U)} = f(u)$ Interior $\overline{f(U)}$ is not empty and thus Interior $\overline{f(U)} \neq \emptyset$. □

A space is called *Polish* if is is completely metrizable and second countable.

It is not too hard to see that the product space $\mathbb{Z}^{\mathbb{N}}$ is a totally disconnected and indeed zero dimensional Polish topological group. In particular, $\mathbb{Z}^{\mathbb{N}}$ is a Baire space. It is harder to see that $\mathbb{Z}^{\mathbb{N}}$ and the space $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers in its natural topology is homeomorphic to $\mathbb{Z}^{\mathbb{N}}$. Thus there are fairly natural Polis (hence Baire) spaces which appear in unexpected guises.

Corollary 3.5. *A surjective morphism between two Polish topological groups is open.*

Proof. By the Baire Category Theorem a Polish space is a Baire space and thus is inexhaustible. Thus the Open Mapping Theorem A proves the Corollary. □

A space X is called *sigma-compact* or *σ-compact* if is is a countable union of compact subspaces.

Theorem 3.6. (Open Mapping Theorem B) *Let $f: G \to H$ be a surjective morphism of topological groups and assume that*
(i) *G is locally compact and σ-compact.*
(ii) *H is inexhaustible and Hausdorff.*
Then f is open.

Proof . If G is σ-compact so is $G/\ker f$, and f is open iff the natural bijective morphism $F: G/\ker f \to H$ of 1.14 is open. It is therefore no restriction of generality to assume that f is bijective as we do now; we have to show that f^{-1} is continuous. Since G is σ-compact there is a countable family of compact subsets $K_n \subseteq G$ such that $G = \bigcup_{n\in\mathbb{N}} K_n$. Then $H = \bigcup_{n\in\mathbb{N}} f(K_n)$ and each$f(K_n)$ is compact; since H is Hausdorff, each $f(K_n)$ is closed in H, and since H is inexhaustible, there is an $n \in N$ and an $h \in H$ such that $h \in V \stackrel{\text{def}}{=} \text{Interior}\, f(K_n)$. Then $U \stackrel{\text{def}}{=} f^{-1}(V)$ is a nonempty open subset of G contained in K_n. Since H is Hausdorff and f is bijective continuous and bijective, G is Hausdorff. Hence K_n is compact Hausdorff and thus $f|K_n : K_n \to f(K_n)$ is a homeomorphism. Then $f|U : U \to V$ is a homeomorphism. Hence $f^{-1}|V : V \to U$ is continuous and therefore f^{-1} is continuous on the open identity neighborhood $h^{-1}V$ of H. Then f^{-1} is continuous by 1.13(a). □

Corollary 3.8. (Open Mapping Theorem for Locally Compact Groups) *A surjective morphism $f: G \to H$ of locally compact topological groups is open if G is σ-compact.*

Proof . By the Baire Category Theorem a locally compact group H is inexhaustible. Then Theorem 3.7 yields the assertion. □

The identity map $\mathbb{R}_d \to \mathbb{R}$ from the discrete additive group of real numbers to the same group with its natural topology is a bijective morphism between first countable locally compact groups which fails to be open.

Exercise E3.4. Prove the following proposition
In a topological group, each of the following conditions implies the next.

(i) *G is locally compact and connected.*
(ii) *G is compactly generated.*
(iii) *G is σ-compact.*

[Hint. (i)⇒(ii) Let U be a compact identity neighborhood. Then $\langle U\rangle$ is an open, hence closed subgroup.

(ii)⇒(iii) If C is a compact subset with $\langle C\rangle = G$, then $K = C \cup C^{-1}$ is a symmetric compact subset such that $G = \bigcup_{n\in\mathbb{N}} K^n$ and each K^n is compact.]

The discrete group $(\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}$ is countable hence σ-compact but is not finitely generated and thus not compactly generated. The discrete group $\mathbb{Z}$ is cyclic, hence compactly generated but is not connected.

Theorem 3.9. (Open Mapping Theorem C) *Let $f: G \to H$ be a surjective morphism of Hausdorff topological groups and assume that*

(i) *G is the additive group of a first countable complete topological vector space.*
(ii) *All maps $p_n: H \to H$, $n \in \mathbb{N}$, $p_n(h) = h^n$, are homeomorphisms, and H is inexhaustible.*

Then f is open.

Proof . By Theorem 3.2 it suffices to show that for each $U \in \mathfrak{U}_G$ the set $\overline{f(U)}$ has inner points in H.

Thus let $U \in \mathfrak{U}_G$. Since G is the additive group of a topological vector space, for each $g \in G$ there is a natural number n such that $\frac{1}{n}\cdot g \in U$. Hence $G = U \cup 2\cdot U \cup 3\cdot U \cup \cdots$. Since f is surjective, $H = f(G) = f(U) \cup f(2\cdot U) \cup f(3\cdot U) \cup \cdots$. Now $f(n\cdot U) = p_n\big(f(U)\big) \subseteq p_n\big(\overline{f(U)}\big)$. Since p_n is a closed function by (ii), the set $p_n\big(\overline{f(U)}\big)$ is closed. since H is inexhaustible, there is an n such that $p_n\big(\overline{f(U)}\big)$ has inner points. Since $p_n\colon H \to H$ is a homeomorphism, $\overline{f(U)}$ has inner points. □

If H is the additive group of a topological vector space, then (ii) is automatically satisfied and we get a much acclaimed theorem of functional analysis

Corollary 3.10. (Open Mapping Theorem for Linear Operators) *If G and H are topological vector spaces and $f\colon G \to H$ is a continuous surjective linear map, then it is auomatically open if G is first countable and complete and H is inexhaustible.* □

In particular, *a surjective bounded linear operator between completely metrizable topological vector spaces is open.*

Such vector spaces are called *Frechet spaces.* In particular, every Banach space is a Frechet space, and thus *a bounded surjective operator between Banach spaces is open.*

It may serve a useful purpose to give a brief overview of various countability conditions.

Let $X = (X, \mathcal{O})$ be a topological space.

	Terminology	Property
(1)	First Axiom of Countability	Each neighborhood filter has a countable basis
(2)	Second Axiom of Countability	The topology $\mathcal{O}$ has a countable basis
(S)	separable	X has a countable dense subset $\{x_1, x_2, \ldots\}$
(K)	σ-compakt	X is a countable union $K_1 \cup K_2 \cup \cdots$ of compact subsets

We observe the following implications:

$$(2) \Rightarrow [(1) \text{ and } (S)], \qquad [(1) \text{ and } (S)] \not\Rightarrow (2)$$

$$[(1) \text{ and } (S) \text{ and } X \text{ is a topological group }] \Rightarrow (2)$$

$$X \text{ metric } \Rightarrow (1), \qquad [X \text{ is metric and } (S)] \Rightarrow (2)$$

$$[X \text{ is a topological group and } (K)] \Rightarrow [(1) \Leftrightarrow (2)].$$

$$[X \text{ is metric complete and } (2)] \Leftrightarrow [X \text{ is Polish}]$$

$$[X \text{ is metric complete}] \Rightarrow [X \text{ is a Baire space}]$$

$$[X \text{ is locally compact}] \Rightarrow [X \text{ is a Baire space}]$$

$$[X \text{ is a Baire space}] \Rightarrow [X \text{ is inexhaustible}]$$

Closed Graph Theorems

There is a class of theorems coming along with the open mapping theorems, namely, the so called closed graph theorems.

Definition 3.11. Let $f: X \to Y$ be a function between topological spaces. The *graph* of f is the subspace $\mathrm{Gr}(f) = \{(x, f(x)): x \in X\}$ of the product $X \times Y$. The graph inherits the subspace topology from the product topology.

If X and Y are groups, then the graph is easily seen to be a subgroup if and only if f is a morphism of groups. □

Remark 3.11. (i) There is a bijective function

$$p: \mathrm{Gr}(f) \to X \quad \text{given by} \quad p\big(x, f(x)\big) = x$$

(i.e., the restriction of the projection $X \times Y \to X$ to $\mathrm{Gr}(f)$). The inverse function of p is $f_0: X \to \mathrm{Gr}(f)$ given by $f_0(x) = (x, f(x))$.

(ii) As a restriction of the continuous projection, p is always continuous. Its inverse function f is continuous.

(iii) If $\mathrm{pr}_Y: X \times Y \to Y$ denotes the second projection, then

$$f = (pr_Y|\mathrm{Gr}(f)) \circ f_0$$

is continuous if and only if f_0 is continuous. □

As a consequence of 3.11 we know that for continuous functions f, the graph of f is homeomorphic to X.

We collect some of these insights in the following remark

Remark 3.12. For a function $f: X \to Y$ between topological spaces the following statements are equivalent:

(1) f ist continuous.
(2) The continuous map $p: \mathrm{Gr}(f) \to X$, $p\big(x, f(x)\big) = x$ is open. □

From an earlier exercise we review the following remark

Remark 3.13. If $f: X \to Y$ is a continuous function between topological spaces and if Y is aHausdorff space, then the graph $\mathrm{Gr}(f)$ of f is closed in $X \times Y$.

Proof. We must show that the complement $(X \times Y) \setminus \mathrm{Gr}(f)$ is open. Let $(x, y) \notin \mathrm{Gr}(f)$, i.e., $y \neq f(x)$. Then there exist open disjoint neighborhoods U_y of y and V_x of $f(x)$ since Y satisfies T_2. Because f is continuous, there is an open neighborhood

U_x of x such that $f(U_x) \subseteq V_x$. As a consequence, $f(U_x) \cap U_y = \emptyset$. We claim that $(U_x \times U_y) \cap Gr(f) \neq \emptyset$. If not, there is an $(x', y') \in (U_x \times U_y) \cap \mathrm{Gr}(f)$. Then $(x', y') \in \mathrm{Gr}(f)$ implies $y' = f(x')$ and so $x' \in U_x$ and $f(x') = y' \in U_y$. This is impossible because of $f(U_x) \cap U_y = \emptyset$. Thus we have an open neighborhood $U_x \times U_y$ of $(x, y) \in X \times Y$ that does not meet $\mathrm{Gr}(f)$. This proves the claim. □

This suggests the reverse question when, conversely, the closedness of the Graphen $\mathrm{Gr}(f)$ implies the continuity of f. In the exercises we proved the following example of such a theorem:

Proposition 3.14. *Let $f\colon X \to Y$ be a function of a topological space X into a compact Hausdorff space Y. Then the following tw statements are equivalent*

(1) *f is continuous.*
(2) *$\mathrm{Gr}(f)$ is closed.*

Proof. (1) ⇒ (2) was shown in 3.13.

(2) ⇒ (1). Let $x \in X$ and V an open neighborhood of $f(x)$. We have to find an open neighborhood U of x such that $f(U) \subseteq V$. Suppose such a U does not exist. Then $\{U \setminus V : U \in \mathfrak{U}(x)\}$ is a filter basis in the compact space $Y \setminus V$. Hence there is a

$$y \in \bigcap_{U \in \mathcal{U}(x)} \overline{f(U)} \setminus V.$$

Since $y \notin V$ this would imply $y \neq f(x)$, i.e., $(x, y) \notin \mathrm{Gr}(f)$. On the other side let $U \times W$ be an arbitrary basic neighborhood of (x, y) in $X \times Y$. Then $y \in \overline{f(U)}$ implies $W \cap f(U) \neq \emptyset$, since W is an open neighborhood of y. Now let $w \in W \cap f(U)$; then $w = f(u) \in W$, where $u \in U$. So $(u, w) \in (U \times W) \cap \mathrm{Gr}(f)$. Hence $(x, y) \in \overline{\mathrm{Gr}(f)}$. By (2), the graph is closed, and so $(x, y) \in \mathrm{Gr}(f)$, contradicting $(x, y) \notin \mathrm{Gr}(f)$ which we showed above. □

Now we come to a closed graph theorem that is attached to the Open Mapping Theorems we discussed.

Theorem 3.15. *Closed Graph Theorem For a morphism $f\colon G \to H$ of topological groups we assume at least one of the following hypotheses:*

(I) G and H are locally compact and σ-compact.
(II) G and H are Polish.
(III) G and H are complete Hausdorff topological vektor spaces satisfying the First Axiom of Countability.

Then the following conditions are equivalent:

(1) f is continuous.
(2) $\mathrm{Gr}(f)$ is closed.

Proof. In all cases, G and H are Hausdorff spaces. Then $G \times H$ and so $\mathrm{Gr}(f)$ are Hausdorff spaces. We have to show that (2) implies (1).

(I) If G and H are locally compact, then $G \times H$ is locally compact. If $\mathrm{Gr}(f)$ is closed, then $\mathrm{Gr}(f)$ is locally compact as well. The same is true for σ-compactness. Then $p: \mathrm{Gr}(f) \to G$ is a surjective momorphism of topological groups between locally compact groups with a σ-compact domain. Then by the Open Mapping Theorem 3.8 for Locally Compact Groups, p is open, and so f is continuous by 3.12.

(II) If G und H re Polish, then this is the case for $G \times H$ and for $\mathrm{Gr}(f)$ by (2). Then p is open by the Open Mapping Theorem 3.5 for Polish Spaces. Again f is continuous by 3.12.

(III) If G and H are topological vector spaces, then $G \times H$ is a topological vector spac. Since f is a morphism of topological groups betwseen reall topological vektor spaces, f is linear, and then $\mathrm{Gr}(f)$ is a vector subspace. By (III), the hypotheses of the Open Mapping Theorem 3.10 for Linear Operators are satisfied, and so p is open. Consequencetly f is continuous, and the Theorem is proved. □

The Second Isomorphism Theorem

If H is a closed subgroup and N a closed normal subgroup of a topological group G, then there is a natural bijective continuous morphism $\beta: H/(H \cap N) \to G/N$, $\beta\big(h(H \cap N)\big) = hN$. Whenever β is an isomorphism of topological groups one refers to this statement as the Second Isomorphism Theorem of Group Theory. Unfortunately, in general, the Second Isomorphism Theorem is not guaranteed for topological groups without additional hypotheses. However, if an Open Mapping Theorem applies to β, the Second Isomorphy Theorem follows. Let $f: H \to HN/N$, $f(h) = hN$. The canonical decomposition theorem yields the commuting diagram

$$\begin{array}{ccc} H & \xrightarrow{\quad f \quad} & HN/N \\ {\scriptstyle \mathrm{quot}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ H/(H \cap N) & \xrightarrow[\quad \beta \quad]{} & HN/N, \end{array}$$

and it suffices to have an Open Mapping Theorem for f. We assume all groups to be Hausdorff.

Theorem 3.16. (The Second Isomorphism Theorem for Pro-Lie groups) *The following conditions are sufficient for the natural morphism*

$$h(H \cap N) \mapsto hN : H/(H \cap N) \to HN/N$$

to be an isomorphism of topological groups:

(i) H is locally compact and σ-compact and HN is inexhaustible.
(ii) H and HN/N are Polish.
(iii) $H/(H \cap N)$ is compact.

Proof . (i) If HN is inexhaustible, then so is HN/N; ideed let HN be the union of a countable set of closed substs A_n, then NH is the union of the closed sets

$q_n^{-1}(A_n)$ where $q_n: HN \to HN/N$ is the quotient map. So for one n there is a nonempty open set U of HN contained in $q^{-1}(A_n)$ and so $q(U)$ is a nonempty open subset of A_n as q is an open map. Now the Open Mapping Theorem 3.6 for Locally Compact Groups applies.

(ii) This follows via the Open Mapping Theorem 3.5 for Polish groups.

(iii) Is clear since bijective continuous functions from a compact space to a Hausdorff space is always a closed map and thus is a homeomorphism. □

Chapter 4
The Fundamental Theorem on Compact Groups

For the moment, let G denote a compact Hausdorff space. Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$. An element μ of the topological dual E' of the Banach space $E = C(G, \mathbb{K})$ or is a ($\mathbb{K}$-valued) *integral* or *measure* on G. (It is not uncommon in our context to use the words "integral" and "measure" synonymously; the eventual justification is, as is usual in the case of such an equivocation, a theorem; here it is the Riesz Representation Theorem of measure theory.) The number $\mu(f)$ is also written $\langle \mu, f \rangle$ or indeed $\int f\,d\mu = \int_G f(g)\,d\mu(g)$. It is not our task here to develop or review measure theory in full. What we need is the uniqueness and existence of one and only one particular measure on a compact group G which is familiar from the elementary theory of Fourier series as Lebesgue measure on the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The formulation of the existence (and uniqueness theorem) is easily understood. We shall be satisfied by indicating a proof through a sequence of exercises.

Definition 4.1. Let G denote a compact group. For any function $f: G \to \mathbb{K}$ define ${}_gf(x) = f(xg)$. A measure μ is called *invariant* if $\mu({}_gf) = \mu(f)$ for all $g \in G$ and $f \in E = C(G, \mathbb{K})$. It is called a *Haar measure* if it is invariant and *positive*, that is, satisfies $\mu(f) \geq 0$ for all $f \geq 0$. The measure μ is called *normalized* if $\mu(1) = 1$ where 1 also indicates the constant function with value 1. □

Example 4.2. If $p: \mathbb{R} \to \mathbb{T}$ denotes the morphism given by $p(t) = t + \mathbb{Z}$ and $C_1(\mathbb{R}, \mathbb{K})$ denotes the Banach space of all continuous functions $f: \mathbb{R} \to \mathbb{K}$ with period 1, then $f \mapsto f \circ p: C(\mathbb{T}, \mathbb{K}) \to C_1(\mathbb{R}, \mathbb{K})$ is an isomorphism of Banach spaces. The measure γ on $\mathbb{T}$ defined by $\gamma(f) = \int_0^1 (f \circ p)(x)\,dx$ with the ordinary Riemann integral on $[0, 1]$ is a normalized Haar measure on $\mathbb{T}$. □

Exercise E4.1. Verify the assertion of Example 4.2. Give a normalized Haar measure on $\mathbb{S}^1$. For $n \in \mathbb{Z}$ define $e_n: \mathbb{T} \to \mathbb{C}$ by $e_n(t + \mathbb{Z}) = e^{2\pi i n t}$. Compute $\gamma(e_j \bar{e}_k)$ for $j, k \in \mathbb{Z}$. □

We now state the Existence and Uniqueness Theorem on Haar Measure. We shall outline one of its numerous proofs in E4.1.

Theorem 4.3. (The Existence and Uniqueness Theorem of Haar measure). For each compact group G there is one and only one normalized Haar measure. $\square$

Exercise E4.2. Use the preceding theorem to show that any Haar measure γ also satisfies the following conditions:
(i) $\int_G f(gt)\,d\gamma(t) = \gamma(f)$ for all $g \in G$ and $f \in C(G, \mathbb{K})$.
(ii) $\int_G f(t^{-1})\,d\gamma(t) = \gamma(f)$ for all $f \in C(G, \mathbb{K})$. $\square$

Definition 4.4. We shall use the notation $\gamma \in C(G, \mathbb{K})'$ for the unique normalized Haar measure, and we shall also write $\gamma(f) = \int_G f(g)\,dg$. $\square$

Consequences of Haar Measure

Theorem 4.5. (Weyl's Trick). Let G be a compact group and E a G-module which is, at the same time, a Hilbert space. Then there is a scalar product relative to which all operators $\pi(g)$ are unitary.

Specifically, if $(\bullet \mid \bullet)$ is the given scalar product on E, then

$$\langle x \mid y \rangle = \int_G (gx \mid gy)\,dg \tag{1}$$

defines a scalar product such that

$$M^{-2}(x \mid x) \le \langle x \mid x \rangle \le M^2(x \mid x) \tag{2}$$

with

$$M = \sup\{\sqrt{(gx \mid gx)} \mid g \in G,\ (x \mid x) \le 1\}, \tag{3}$$

and that

$$\langle gx \mid gy \rangle = \langle x \mid y \rangle \quad \text{for all} \quad x,\, y \in E,\ g \in G. \tag{4}$$

Proof . For each x, $y \in E$ the integral on the right side of (1) is well-defined, is linear in x and conjugate linear in y. Since Haar measure is positive, the information $(gx \mid gx) \ge 0$ yields $\langle x \mid x \rangle \ge 0$. The positive number M in (3) is well-defined since $\pi(G)$ is compact in $\mathrm{Hom}(E, E)$. Then $\langle x \mid x \rangle \le \int_G M^2(x \mid x)\,dg = M^2(x \mid x)$ since γ is positive and normalized. Also, $(x \mid x) = (g^{-1}gx \mid g^{-1}gx) \le M^2(gx \mid gx)$, whence $\langle x \mid x \rangle \ge \int_G M^{-2}(x \mid x)\,dg = M^{-2}(x \mid x)$. This proves (2) and thus also the fact that $\langle \bullet \mid \bullet \rangle$ is positive definite, that is, a scalar product. Finally, let $h \in G$; then $\langle hx \mid hy \rangle = \int_G (ghx \mid ghy)\,dg = \int_G (gx \mid gy)\,dg = \langle x \mid y \rangle$ by the invariance of γ. $\square$

The idea of the construction is that for each $g \in G$ we obtain a scalar product $(x, y) \mapsto (gx \mid gy)$. The invariant scalar product is the "average" or "expectation" of this family with respect to the probability measure γ.

Definition 4.6. If G is a topological group, then a *Hilbert G-module* is a Hilbert space E and a G-module such that all operators $\pi(g)$ are unitary, that is, such that

$$(gx \mid gy) = (x \mid y) \quad \text{for all} \quad x,\, y \in E,\ g \in G. \qquad \square$$

The following Lemma will show that in every Hilbert G-module of a locally compact group the action$(g, x) \mapsto gx\colon G \times E \to E$ is continuous.

Lemma 4.7. *Assume that E is a G-module for a topological group G and that $\pi\colon G \to \mathcal{L}_p(E)$ is the associated representation* (*see* 2.2(ii))*. If E is a Baire space, then for every compact subspace K of G the set $\pi(K) \subseteq \mathrm{Hom}(E, E)$ is equicontinuous at* 0*, that is, for any neighborhood V of* 0 *in E there is a neighborhood U of* 0 *such that $KU \subseteq V$.*

As a consequence, if G is locally compact, the function

$$(g, x) \mapsto gx\colon G \times E \to E$$

is continuous.

Proof. First step: Given V we find a closed 0-neighborhood W with $W - W \subseteq V$ and $[0,1]{\cdot}W \subseteq W$. Notice that also the interior, Interior W, of W is star-shaped, that is, satisfies $[0,1]{\cdot}\,\mathrm{Interior}\,W = \mathrm{Interior}\,W$. Next we consider

$$C = \bigcap_{g \in K} g^{-1}W.$$

Since K is compact, Kx is compact for any $x \in E$ and thus, as $Kx \subseteq E = \bigcup_{n\in\mathbb{N}} n{\cdot}\,\mathrm{Interior}\,W$ and the $n{\cdot}W$ form an ascending family, we find an $n \in N$ with $K \cdot x \subseteq n{\cdot}W$, that is, with $x \in \bigcap_{g\in K} n{\cdot}g^{-1}W$. Hence for each $x \in E$ there is a natural number n such that $x \in n{\cdot}C$. Therefore

$$E = \bigcup_{n\in\mathbb{N}} n{\cdot}C,$$

where all sets $n{\cdot}C$ are closed. But E is a Baire space, and so for some $n \in \mathbb{N}$, the set $n{\cdot}C$ has interior points, and since $x \mapsto n{\cdot}x$ is a homeomorphism of E, the set C itself has an interior point c. Now for each $g \in K$ we find $g(C - c) \subseteq W - W \subseteq V$. But $U = C - c$ is a neighborhood of 0, as $KU \subseteq V$, our first claim is proved.

Second step: For a proof of the continuity of the function $\alpha = \big((g, x) \mapsto gx\big)$: $G \times E \to E$, it suffices to show the continuity of α at the point $(\mathbf{1}, 0)$. To see this it suffices to note that for fixed $h \in G$ and fixed $y \in E$ the d ifference $\alpha(g, x) - \alpha(h, y) = gx - hy = h\big(h^{-1}g(x - y) + (h^{-1}gy - y)\big) = \pi(h)\big(\alpha(h^{-1}g, x - y) + (h^{-1}gy - y)\big)$ falls into any given neighborhood of 0 as soon as $h^{-1}g$ is close enough to $\mathbf{1}$ and the difference $x - y$ is close enough to zero, because α is continuous at $(\mathbf{1}, 0)$, because $\pi(h)$ is continuous and because $k \mapsto ky\colon G \to E$ is continuous by by the definition of the topology of pointwise convergence.

Third step: We now assume that G is locally compact and show that α is continuous at $(\mathbf{1}, 0)$. For this purpose it suffices to know that for a compact neighborhood K of $\mathbf{1}$ in G the set $\pi(K) \subseteq \mathrm{Hom}(E, E)$ is equicontinuous; for then any neighborhood V of 0 yields a neighborhood U of 0 in E with $\alpha(K \times U) = \pi(K)(U) \subseteq V$. This completes the proof of the second claim. □

According to the above theorem, if G is a compact group, and E is a G-module which is at the same time a Banach space, the compact group G acts on E; that is, $(g, x) \mapsto gx : G \times E \to E$ is continuous.

Example 4.8. Let G be a compact group. Set $E = C(G, \mathbb{K})$; then E is a Banach space with respect to the sup-norm given by $\|f\| = \sup_{t \in G} |f(t)|$. We define ${}_gf = \pi(g)(f)$ by ${}_gf(t) = f(tg)$. Then $\pi\colon G \to \mathcal{L}_p(E)$ is a faithful (that is, injective) representation, and G acts on E.

Proof . We note $|f_1(tg_1) - f_2(tg_2)| \leq |f_1(tg_1) - f_1(tg_2)| + |f_1(tg_2) - f_2(tg_2)| \leq |f_1(tg_1) - f_1(tg_2)| + \|f_1 - f_2\|$. Since G is compact, f_1 is uniformly continuous. Hence the first summand is small if g_1 and g_2 are close. The second summand is small if f_1 and f_2 are close in E. This shows that $(g, f) \mapsto {}_gf\colon G \times E \to E$ is continuous. It is straightforward to verify that this is a linear action. Finally $\pi(g) = \mathrm{id}_E$ is tantamount to $f(tg) = f(t)$ for all $t \in G$ and all $f \in C(G, \mathbb{K})$. Since the continuous functions separate the points, taking $t = 1$ we conclude $g = 1$. □

In Example 4.8, under the special hypotheses present, we have verified the conclusion of Lemma 4.7 directly.

By Weyl's Trick 4.5, for compact G, it is never any true loss of generality to assume for a G-module on a Hilbert space that E is a Hilbert module. Every finite dimensional $\mathbb{K}$-vector space is a Hilbert space (in many ways). Thus, in particular, *every representation of a compact group on a finite dimensional* $\mathbb{K}$-*vector space may be assumed to be unitary.*

Hilbert modules are the crucial type of G-modules for compact groups G as we shall see presently. For the moment, let us observe, that every compact group G has at least one faithful Hilbert module.

Example 4.9. Let G be a compact group and $\mathcal{H}_0$ the vector space $C(G, \mathbb{K})$ equipped with the scalar product

$$(f_1 \mid f_2) = \gamma(f_1\overline{f_2}) = \int_G f_1(g)\overline{f_2(g)}\, dg.$$

Indeed the function $(f_1, f_2) \mapsto (f_1 \mid f_2)$ is linear in the first argument, conjugate linear in the second, and $(f \mid f) = \gamma(f\overline{f}) \geq 0$ since γ is positive. Also, if $f \neq 0$, then there is a $g \in G$ with $f(g) \neq 0$. Then the open set $U = \{t \in G \mid (f\overline{f})(t) > 0\}$ contains g, hence is nonempty. The relation $(f \mid f) = 0$ would therefore imply that U does not meet the support of γ, which is G—an impossibility. Hence the scalar product is positive definite and $\mathcal{H}_0$ is a pre-Hilbert space. Its completion is a Hilbert space $\mathcal{H}$, also called $\mathrm{L}^2(G,\mathbb{K})$.

The translation operators $\pi(g)$ given by $\pi(g)(f) = {}_gf$ are unitary since $(\pi(g)f \mid \pi(g)f) = \int_G f(tg)\overline{f(tg)}\,dt = \int_g f(t)\overline{f(t)}\,dt = (f \mid f)$ by invariance. Every unitary operator on a pre-Hilbert space $\mathcal{H}_0$ extends uniquely to a unitary operator on its completion $\mathcal{H}$, and we may denote this extension with the same symbol $\pi(g)$.

The space $\mathcal{L}(\mathcal{H})$ of bounded operators on the Hilbert space $\mathcal{H}$ is a C*-algebra and $\mathcal{U}(\mathcal{H}) = \mathcal{U}\big(\mathcal{L}(\mathcal{H})\big)$ denotes its unitary group. Then $\pi\colon G \to \mathcal{U}(\mathcal{H})$ is a morphism of groups. We claim that it is continuous with respect to the strong operator topology, that is, $g \mapsto {}_gf\colon G \to \mathcal{H}$ is continuous for each $f \in \mathcal{H}$. Let $\varepsilon > 0$ and let $f_0 \in C(G,\mathbb{K})$ be such that $\|f - f_0\|_2 < \varepsilon$ where $\|f\|_2^2 = (f \mid f)$. Then $\|{}_gf - {}_hf\|_2 \leq \|{}_gf - {}_gf_0\|_2 + \|{}_gf_0 - {}_hf_0\|_2 + \|{}_hf_0 - {}_hf\|_2 = \|{}_gf_0 - {}_hf_0\|_2 + 2\|f - f_0\|_2 < \|{}_gf_0 - {}_hf_0\|_2 + 2\varepsilon$ in view of the fact that $\pi(g)$ is unitary. But $\|{}_gf_0 - {}_hf_0\|_2 \leq \|{}_gf_0 - {}_hf_0\|_\infty$ where $\|f_0\|_\infty$ is the sup-norm $\sup_{g\in G}|f_0(g)|$ for a continuous function f_0. By Example 4.8 the function $g \mapsto {}_gf$ is continuous with respect to the sup-norm; hence $\|{}_gf_0 - {}_hf_0\|_\infty$ can be made less than ε for g close enough to h. For these g and h we then have $\|{}_gf - {}_hf\|_2 < 3\varepsilon$. This shows the desired continuity. Since $\pi(g) = \mathrm{id}_{\mathcal{H}}$ implies $\pi(g)|\mathcal{H}_0 = \mathrm{id}_{\mathcal{H}_0}$ and this latter relation already implies $g = \mathbf{1}$ by Example 4.8, the representation π is injective. Thus $\mathrm{L}^2(G,\mathbb{K})$ *is a faithful Hilbert module.* It is called the regular G-module and the unitary representation $\pi\colon G \to \mathcal{U}\big(\mathrm{L}^2(G,\mathbb{K})\big)$ is called the *regular representation.* ☐

For the record we write:

Remark 4.10. Every compact group possesses faithful unitary representations and faithful Hilbert modules. ☐

The Main Theorem on Hilbert Modules for Compact Groups

We consider a Hilbert space $\mathcal{H}$. A *sesquilinear form* is a function $B\colon \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ that is linear in the first and conjugate linear in the second argument, and that is bounded in the sense that there is a constant M such that $|B(x,y)| \leq M\|x\|\cdot\|y\|$ for all x, $y \in \mathcal{H}$. If T is a bounded linear operator on $\mathcal{H}$, then $B(x,y) = (Tx \mid y)$ defines a sesquilinear form with $M = \|T\|$ in view of the Inequality of Cauchy and Schwarz saying that $|(x \mid y)| \leq \|x\|\cdot\|y\|$. (For our purposes we included continuity in the definition of sesquilinearity.)

Lemma 4.11. *If B is a sesquilinear form, then there exists a unique bounded operator T of $\mathcal{H}$ such that $\|T\| \leq M$ and that $B(x,y) = (Tx \mid y)$.*

Proof . Exercise. ☐

Exercise E4.3. Prove Lemma 4.11.

[Hint. Fix $x \in \mathcal{H}$. The function $y \mapsto B(x,y)$ is a bounded conjugate linear form on $\mathcal{H}$. Hence there is a unique element $Tx \in \mathcal{H}$ such that $B(x,y) = (Tx \mid y)$ by the elementary Riesz Representation Theorem for Hilbert spaces. The function $T = (x \mapsto Tx)\colon \mathcal{H} \to \mathcal{H}$ is linear. Use $|B(x,y)| \le M\|x\|\cdot\|y\|$ to deduce $\|T\| \le M$.]□

Lemma 4.12. *Let G denote a compact group and T a bounded operator on a Hilbert G-module E. Then there is a unique bounded operator $\widetilde{T}$ on E with $\|\widetilde{T}\| \le \|T\|$ such that*

$$(\widetilde{T}x \mid y) = \int_G (Tgx \mid gy)\,dg = \int_G (\pi(g)^{-1}T\pi(g)(x) \mid y)\,dg. \tag{5}$$

Proof . Since π is a unitary representation, $\pi(g)^* = \pi(g)^{-1}$ and so the last two integrals in (5) are equal. The prescription $B(x,y) = \int_G (Tgx \mid gy)\,dg$ defines a function B which is linear in x and conjugate linear in y. Because

$$|(Tgx \mid gy)| \le \|Tgx\|\cdot\|gy\| = \|T\|\cdot\|gx\|\cdot\|gy\| = \|T\|\cdot\|x\|\cdot\|y\|$$

(as G acts unitarily on $\mathcal{H}$!) we obtain the estimate $|B(x,y)| \le \int_G \|T\|\cdot\|x\|\cdot\|y\|\,dg = \|T\|\cdot\|x\|\cdot\|y\|$. Hence B is a sesquilinear form, and so by Lemma 4.11, there is a bounded operator $\widetilde{T}$ with $B(x,y) = (\widetilde{T}x \mid y)$ and $\|\widetilde{T}\| \le \|T\|$. □

In any ring R, the *commutant* $\mathcal{C}(X)$ (or, in semigroup and group theory equivalently called the *centralizer* $Z(X,R)$) of a subset $X \subseteq R$ is the set of all elements $r \in R$ with $xr = rx$ for all $x \in X$. Using integration of no more than $\mathbb{K}$-valued functions, we have created the operator

$$\widetilde{T} = \int_G \pi(g)^{-1}T\pi(g)\,dg,$$

where the integral indicates an averaging over the conjugates $\pi(g)^{-1}T\pi(g)$ of T. It is clear that the averaging self-map $T \mapsto \widetilde{T}$ of $\mathrm{Hom}(\mathcal{H},\mathcal{H})$ is linear and bounded. Its significance is that its image is exactly the commutant $\mathcal{C}\big(\pi(G)\big)$ of $\pi(G)$ in $\mathrm{Hom}(\mathcal{H},\mathcal{H})$. Thus it is the set of all bounded operators S on $\mathcal{H}$ satisfying $S\pi(g) = \pi(g)S$. This is tantamount to saying that $S(gx) = g(Sx)$ for all $g \in G$ and $x \in \mathcal{H}$. Such operators are also called *G-module endomorphisms* or *intertwining operators*. In the present context the commutant is sometimes denoted also by $\mathrm{Hom}_G(\mathcal{H},\mathcal{H})$.

Lemma 4.13. *The following statements are equivalent for an operator S of $\mathcal{H}$:*

(1) $S \in \mathrm{Hom}_G(\mathcal{H},\mathcal{H})$.
(2) $S = \widetilde{S}$.
(3) *There is an operator T such that $S = \widetilde{T}$.*

Proof . (1)⇒(2) By definition, $(\widetilde{S}x \mid y) = \int_G (Sgx \mid gy)\,dg$. By (1) we know $Sgx = gSx$, and since $\mathcal{H}$ is a unitary G-module, $(Sgx \mid gy) = (gSx \mid gy) =$

$= (Sx \mid y)$. Since γ is normalized, we find $(\widetilde{S}x \mid y) = (Sx \mid y)$ for all x and y in $\mathcal{H}$. This means (2).

(2)⇒(3) Trivial.

(3)⇒(1) Let x and y be arbitrary in $\mathcal{H}$ and $h \in G$. Then

$$\begin{aligned}(Shx \mid y) = (\widetilde{T}hx \mid y) &= \int_G (Tghx \mid gy)\, dg = \int_G \big(Tghx \mid gh(h^{-1}y)\big)\, dg \\ &= \int_G (Tgx \mid gh^{-1}y)\, dg = (\widetilde{T}x \mid h^{-1}y) = (Sx \mid h^{-1}y) = (hSx \mid y)\end{aligned}$$

in view of the invariance of γ and the fact that $\pi(g)^{-1} = \pi(g)^*$. Hence $S\pi(h) = \pi(h)S$ for all $h \in G$ and thus (1) is proved. □

We see easily that $\mathrm{Hom}_G(\mathcal{H}, \mathcal{H})$ is a closed C*-subalgebra of $\mathcal{L}(\mathcal{H})$.

An *orthogonal projection* of $\mathcal{H}$ is an idempotent operator P satisfying $P^* = P$, that is, $(Px \mid y) = (x \mid Py)$ for all $x, y \in \mathcal{H}$. The function $P \mapsto P(\mathcal{H})$ is a bijection from the set of all orthogonal projections of $\mathcal{H}$ to the set of all closed vector subspaces V of $\mathcal{H}$. Indeed every closed vector subspace V has a unique orthogonal complement $V^\perp$ and thus determines a unique orthogonal projection of $\mathcal{H}$ with image V and kernel $V^\perp$.

Definition 4.14. If G is a topological group and E a G-module, then a vector subspace V of E is called a *submodule* if $GV \subseteq V$. Equivalently, V is also called an *invariant subspace.* □

Lemma 4.15. *For a closed vector subspace V of a Hilbert G-module $\mathcal{H}$ and the orthogonal projection P with image V the following statements are equivalent:*

(1) V is a G-submodule.
(2) $P \in \mathrm{Hom}_G(\mathcal{H}, \mathcal{H})$.
(3) $V^\perp$ is a G-submodule.

Proof . (1)⇒(2) Let $x \in \mathcal{H}$; then $x = Px + (1-P)x$ and thus

$$(*) \qquad gx = gPx + g(1-P)x$$

for all $g \in G$. But $Px \in V$ and thus $gPx \in V$ since V is invariant. Since the operator $\pi(g)$ is unitary, it preserves orthogonal complements, and thus $g(1-P)x \in V^\perp$. Then $(*)$ implies $gPx = P(gx)$ $\big($and $g(1-P)x = (1-P)(gx)\big)$.

(2)⇒(3) The kernel of a morphism of G-modules is readily seen to be invariant. Since $V^\perp = \ker P$ and P is a morphism of G-modules, clearly $V^\perp$ is invariant.

(3)⇒(1) Assume that $V^\perp$ is invariant. We have seen in the preceding two steps of the proof that the orthogonal complement $W^\perp$ of any invariant closed vector subspace W of $\mathcal{H}$ is invariant. Now we apply this to $W = V^\perp$. Hence $(V^\perp)^\perp$ is invariant. But $(V^\perp)^\perp = V$, and thus V is invariant. □

Lemma 4.16. *If T is a hermitian (respectively, positive) operator on a Hilbert G-module $\mathcal{H}$, then so is $\widetilde{T}$.*

Proof . For $x, y \in \mathcal{H}$ we have

$$(\widetilde{T}x \mid y) = \int_G (gTg^{-1}x \mid y)\,dg = \int_G (Tg^{-1}x \mid g^{-1}y)\,dg.$$

If $T = T^*$, then $(Tg^{-1}x \mid g^{-1}y) = \overline{(T^*g^{-1}x} \mid g^{-1}y) = (g^{-1}x \mid Tg^{-1}y) = \overline{(Tg^{-1}y \mid g^{-1}x)}$ and thus $(\widetilde{T}x \mid y) = \overline{(\widetilde{T}y \mid x)}$. Hence $\widetilde{T}$ is hermitian. If T is positive, then $\widetilde{T}$ is hermitian by what we just saw, and taking $y = x$ and observing $(Tg^{-1}x \mid g^{-1}x) \geq 0$ we find that $\widetilde{T}$ is positive, too. □

Next we turn to the important class of compact operators. Recall that an operator $T{:}\,V \to V$ on a Banach space is called *compact* if for every bounded subset B of V the image TB is precompact. Equivalently, this says that $\overline{TB}$ is compact, since V is complete.

Lemma 4.17. *If T is a compact operator on a Hilbert G-module $\mathcal{H}$, then $\widetilde{T}$ is also compact.*

Proof . Let B denote the closed unit ball of $\mathcal{H}$. We have to show that $\widetilde{T}B$ is precompact. Since all $\pi(g)$ are unitary, we have $gB = B$ for each $g \in G$. Hence $A \stackrel{\text{def}}{=} \overline{TGB}$ is compact since T is compact. Since the function $(g, x) \mapsto gx{:}\, G \times \mathcal{H} \to \mathcal{H}$ is continuous by Lemma 4.7, the set GA is compact. The closed convex hull K of GA is compact (see Exercise E2.5 below). Now let $y \in \mathcal{H}$ be such that $\mathrm{Re}(x \mid y) \leq 1$ for all $x \in K$. Then $x \in B$ implies $\mathrm{Re}(\widetilde{T}x \mid y) = \int_G \mathrm{Re}(gTg^{-1}x \mid y)\,dg \leq \int_G dg = 1$ since $gTg^{-1}x \in GTGB \subseteq GA \subseteq K$ for all $g \in G$. Hence $\widetilde{T}x$ is contained in every closed real half-space which contains K. From the Theorem of Hahn and Banach we know that a closed convex set is the intersection of all closed real half-spaces which contain it. Hence we conclude $\widetilde{T}x \in K$ and thus $\widetilde{T}B \subseteq K$. This shows that $\widetilde{T}$ is compact. □

It is instructive at this point to be aware of the information used in the preceding proof: the joint continuity of the action proved in 4.7, the precompactness of the convex hull of a precompact set in a Banach space (subsequent Exercise!), the Hahn–Banach Theorem, and of course the compactness of G.

Exercise E4.4. Show that in a Banach space V, the closed convex hull K of a precompact set P is compact.
[Hint. Since V is complete, it suffices to show that K is precompact. Thus let U be any open ball around 0. Since P is precompact, there is a finite set $F \subseteq P$ such that $P \subseteq F + U$. The convex hull S of F is compact (as the image of a finite simplex under an affine map). Hence there is a finite set $F' \subseteq S$ such that

$S \subseteq F' + U$. Now the convex hull of P is contained in the convex set $S + U$, hence in the set $F' + U + U$, and its closure is contained in $F' + U + U + U = F' + 3U$.]□

We can summarize our findings immediately in the following lemma.

Lemma 4.18. *On a nonzero Hilbert G-module $\mathcal{H}$ let x denote any nonzero vector and T the orthogonal projection of $\mathcal{H}$ onto $\mathbb{K}{\cdot}x$. Then $\widetilde{T}$ is a nonzero compact positive operator in* $\mathrm{Hom}_G(\mathcal{H}, \mathcal{H})$.

Proof. This follows from the preceding lemmas in view of the fact that an orthogonal projection onto a one-dimensional subspace $\mathbb{K}{\cdot}x$ is a positive compact operator and that $(Tx \mid x) = \|x\|^2 > 0$, whence $(\widetilde{T}x \mid x) = \int_G (Tg^{-1}x \mid g^{-1}x)\, dg > 0$. □

Now we recall some elementary facts on compact positive operators. Notably, every compact positive nonzero operator T has a positive eigenvalue λ and the eigenspace $\mathcal{H}_\lambda$ is finite dimensional.

Exercise E4.5. Let $\mathcal{H}$ be a Hilbert space and T a nonzero compact positive operator. Show that there is a largest positive eigenvalue λ and that $\mathcal{H}_\lambda$ is finite dimensional.

[Hint. Without loss of generality assume $\|T\| = 1$. Note $\|T\| = \sup\{\|Tx\| \mid \|x\| \leq 1\} = \sup\{\mathrm{Re}(Tx \mid y) \mid \|x\|, \|y\| \leq 1\}$. Since T is positive, $0 \leq (T(x+y) \mid x+y) = (Tx \mid x) - 2\,\mathrm{Re}(Tx \mid y) + (Ty \mid y)$, whence $\mathrm{Re}(Tx \mid y) \leq \frac{1}{2}\big((Tx \mid x) + (Ty \mid y)\big) \leq \max\{(Tx \mid x), (Ty \mid y)\}$. It follows that $\|T\| = \sup\{(Tx \mid x) \mid \|x\| = 1\}$. Now there is a sequence $x_n \in \mathcal{H}$ with $1 - \frac{1}{n} < (Tx_n \mid x_n) \leq 1$ and $\|x_n\| = 1$. Since T is compact there is a subsequence $y_k = x_{n(k)}$ such that $z = \lim_{k \in \mathbb{N}} Ty_k$ exists with $\|z\| = 1$. Now $0 \leq \|Ty_n - y_n\|^2 = \|Ty_n\|^2 - 2{\cdot}(Ty_n \mid y_n) + \|y_n\|^2 \to 1 - 2 + 1 = 0$. Hence $z = \lim y_n$ and $Tz = z$.] □

We now have all the tools for the core theorem on the unitary representations of compact groups.

Theorem 4.19. (The Fundamental Theorem on Unitary Modules) *Every nonzero Hilbert G-module for a compact group G contains a nonzero finite dimensional submodule.*

Proof. By Lemma 4.18 we find a nonzero compact positive operator $\widetilde{T}$ which is invariant by 4.13. But $\widetilde{T}$ has a finite dimensional nonzero eigenspace $\mathcal{H}_\lambda$ for an eigenvalue $\lambda > 0$ by Exercise E4.5 If $\widetilde{T}x = \lambda{\cdot}x$, then $\widetilde{T}gx = g\widetilde{T}x = g(\lambda{\cdot}x) = \lambda{\cdot}gx$. Thus $\mathcal{H}_\lambda$ is the desired submodule. □

Definition 4.20. A G-module E is called *simple* if it is nonzero and $\{0\}$ and E are the only invariant submodules. The corresponding representation of G is called *irreducible*. □

Corollary 4.21. *Every nonzero Hilbert G-module for a compact group G contains a simple nonzero G-module.*

Proof . By the Fundamental Theorem on Unitary Modules 4.20, we may assume that the given module $\mathcal{H}$ is finite dimensional. Every descending chain of nonzero submodules then is finite and thus has a smallest element. It follows that $\mathcal{H}$ has a nonzero minimal submodule which is necessarily simple. □

Corollary 4.22. *Every nonzero Hilbert G-module for a compact group G is a Hilbert space orthogonal sum of finite dimensional simple submodules.*

Proof . Let E be a Hilbert G-module and consider, by virtue of Corollary 4.21 and Zorn's Lemma, a maximal family $\mathcal{F} = \{E_j \mid j \in J\}$ of finite dimensional submodules such that $j \neq k$ in J implies $E_j \perp E_k$. Let H be the closed span of this family (that is, its orthogonal sum). Then H is a G-module. If $H \neq E$, then $H^\perp$ is a nonzero G-module by Lemma 4.15 nonzero simple submodule K. Then $\mathcal{F} \cup \{K\}$ is an orthogonal family of finite dimensional simple submodules which properly enlarges the maximal family $\mathcal{F}$, and this is impossible. Thus $E = H$, and this proves the corollary. □

Definition 4.23. We say that a family $\{E_j \mid j \in J\}$ of G-modules, respectively, the family $\{\pi_j \mid j \in J\}$ of representations *separates the points of G* if for each $g \in G$ with $g \neq \mathbf{1}$ there is a $j \in J$ such that $\pi_j(g) \neq \mathrm{id}_{E_j}$, that is, there is an $x \in E_j$ such that $gx \neq x$. □

Corollary 4.24. *If G is a compact group, then the finite dimensional simple modules separate the points.*

Proof . By Example 4.9, there is a faithful Hilbert G-module E. By Corollary 4.22, the module E is an orthogonal direct sum $\bigoplus_{j\in J} E_j$ of simple finite dimensional submodules E_j. If $g \in G$ and $g \neq \mathbf{1}$, then there is an $x \in E$ such that $gx \neq x$. Writing x as an orthogonal sum $\sum_{j\in J} x_j$ with $x_j \in E_j$ we find at least one index $j \in J$ such that $gx_j \neq x_j$ and this is what we had to show. □

Corollary 4.25. *The orthogonal and the unitary representations $\pi: G \to \mathfrak{O}(n)$, respectively, $\pi: G \to \mathrm{U}(n)$ separate the points of any compact group$G$.*

Proof . By Weyl's Trick 4.5, , for a compact group G, every finite dimensional real representation is orthogonal and every complex finite dimensional representation is unitary for a suitable scalar product. The assertion therefore is a consequence of Corollary 4.24. □

Corollary 4.26. *Every compact group G is isomorphic to a closed subgroup of a product $\prod_{j\in J} \mathfrak{O}(n_j)$ and of a product $\prod_{j\in J} \mathrm{U}(n_j)$ of unitary groups.* □

Index